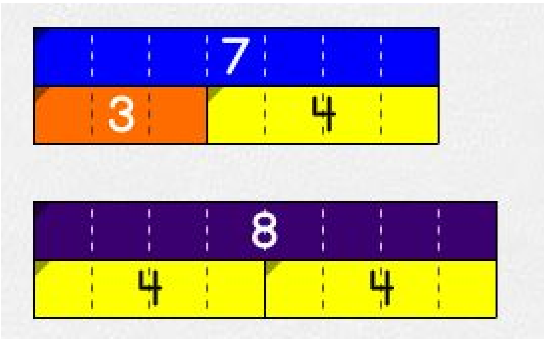
Reasoning and Proving with Relational Rods

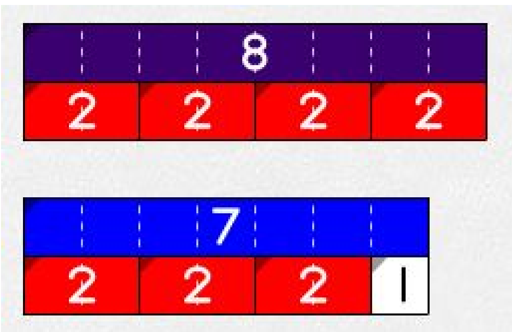
The Ministry of Education in Ontario has invested in the creation of the mathies learning tools because of a strong belief in the power of visual representations to uncover important mathematical concepts. Even concepts of number are revealed and made stronger by relating them to visual representations, like those created within Relational Rods+. This article will provide some elementary and not so elementary examples of how the tool can be used to help students reason about number properties and how connections to algebraic thinking can be made.

Even and Odd Numbers

Ask students to represent a variety of numbers with the Whole Number Rods, like 7, 8, 17 and 24. Ask them whether the numbers are odd or even and how they could use the rods to support their conclusion.



One student might argue that 8 is even because it can be broken into 2 equal pieces of 4, while 7 is odd since it cannot be broken in half by a whole number rod.

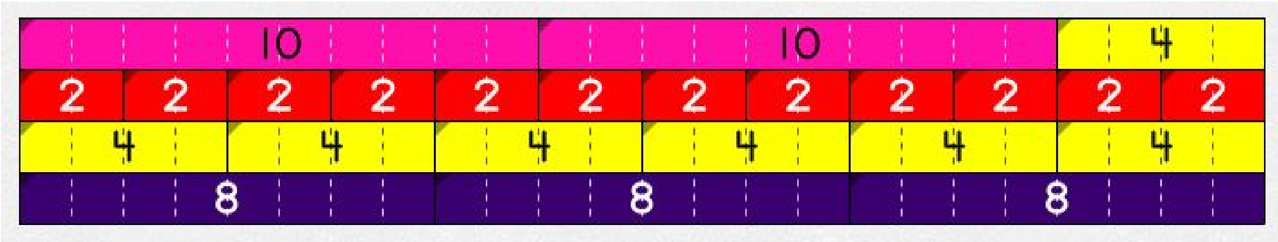


Another student might argue that 8 is even since it can be broken into 2 evenly, with 4 equal pieces of 2, while with 7 there is a pesky 1 left over.

A comparison of the two methods raises some interesting properties of number:

- The commutative property: 2 pieces of 4 and 4 pieces of 2 have the same result
- When dividing into groups of 2, the remainder will be either 0 or 1
- 7 is a near double when thought of as 3 + 4 or as 3 + 3 + 1

The copy feature of the digital tool makes it a lot more fun to create twelve twos by "wailing" on the copy button rather than by dragging each rod out individually.

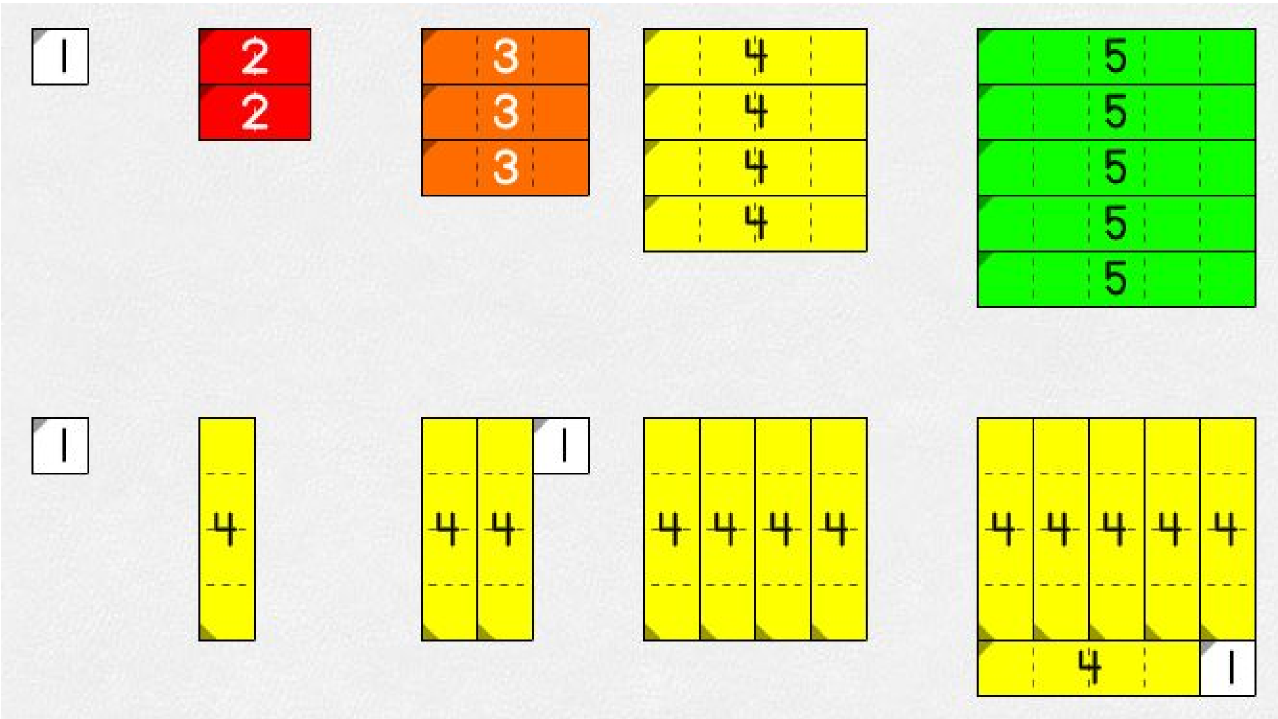


What questions might the representation above elicit after students establish that 24 is even?

Is every multiple of 4 also a multiple of 2? Is every multiple of 8 even? Is every multiple of 8 also a multiple of 16? If 4 goes into a number *k* times, how many times does 8 go into it?

Some examples of conjectures that students could generate or reason about include:

- An odd number plus an odd number is even
- An odd number times an odd number is odd
- The product of two consecutive numbers is even
- Every perfect square is of the form $4k$ or $4k + 1$

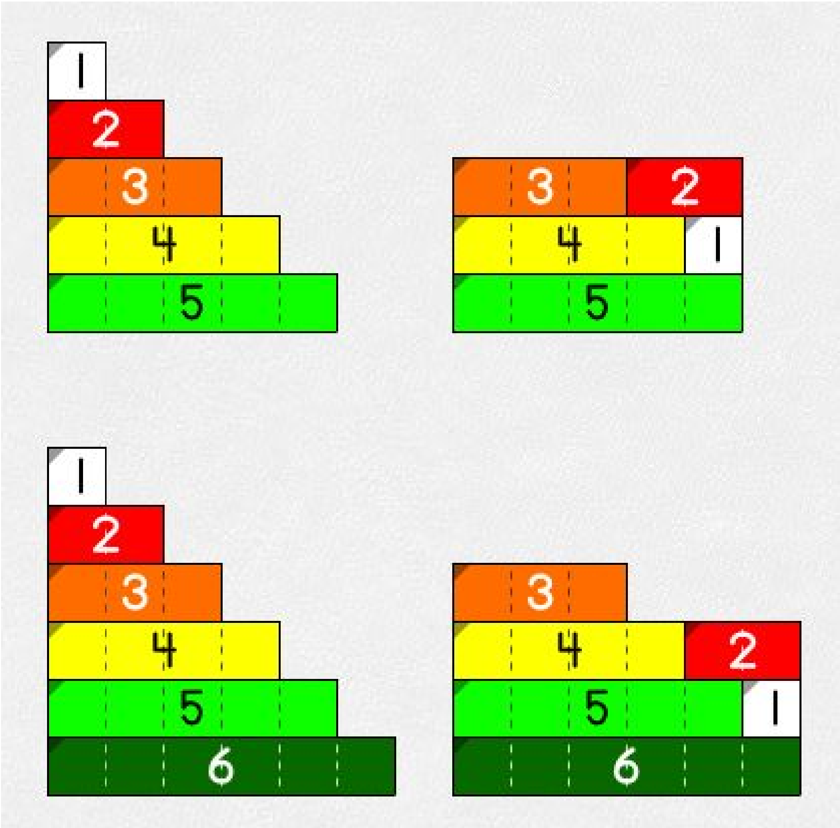


Doesn't this last statement and diagram make you wish we were promising an Algebra Tiles app, where you could create a representation for (2n)(2n) and (2n+1)(2n+1)?

Summing the Natural Numbers

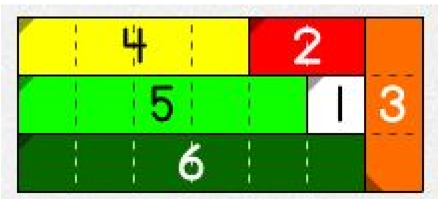
There is no story that inspires kids like the story of how a precocious Gauss embarrassed his teacher by determining the sum of all the natural numbers from 1 to 1000 in seconds. Fortunately, there is no YouTube video of the encounter and students can explore these sums without prejudice.

In the diagrams below, the sums from 1 to 5 and 1 to 6 are rearranged to show that the former is the same as 3 times 5 and the latter is the same as 3 times 6 plus 3.

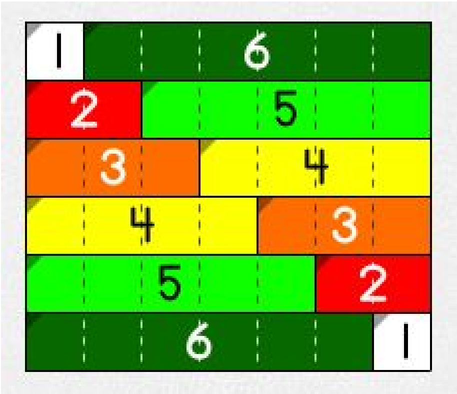


How would the pattern continue? Could this be used to determine the sum from 1 to 1000 quickly?

How might it help to rearrange the last diagram as follows?



Of course, Gauss' trick was more subtle - he must have had the copy button as well!



The diagram above has two copies of 1 + 2 + 3 + 4 + 5 + 6 and forms a 6 by 7 rectangle. This means that 6 times 7 is twice the sum. If 42 is twice the sum, the sum must be 21.

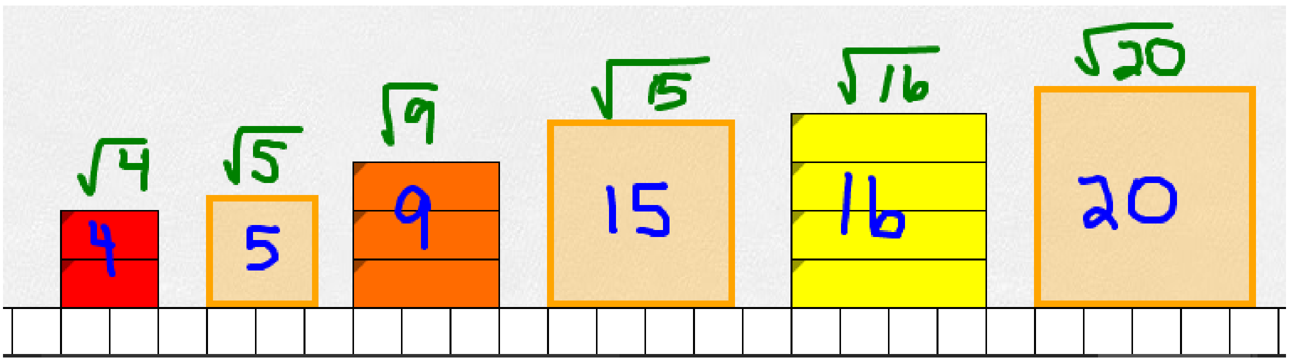
The Paying Attention to Algebraic Reasoning document discusses the importance of using algebra to generalize properties such as this. Here we see that:

$$1 + 2 + 3 + \ldots + n$$
$$= \frac{n(n+1)}{2}$$
$$= \left(\frac{n}{2}\right)(n+1)$$
$$= (n)\left(\frac{n+1}{2}\right)$$

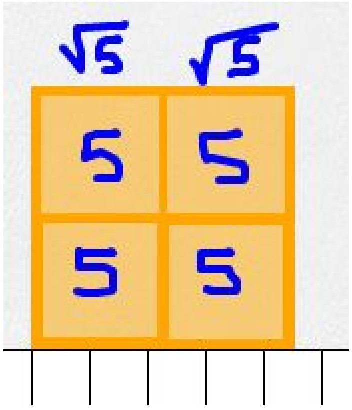
Having older students match the various equivalences above to the corresponding arrangement of the rods above would be most productive.

An Annotation Tool Example

All the recent mathies apps include the ability to annotate, which helps students explain and support their work. If we define the square root of 5 to mean the side length of a square with area 5 (and why wouldn't we?) then we can start to fill in some gaps.



Now we know that converting radicals to mixed and entire form is out of fashion, but how robust would a student's understanding of the meaning of square root, addition and multiplication be to recognize that this:

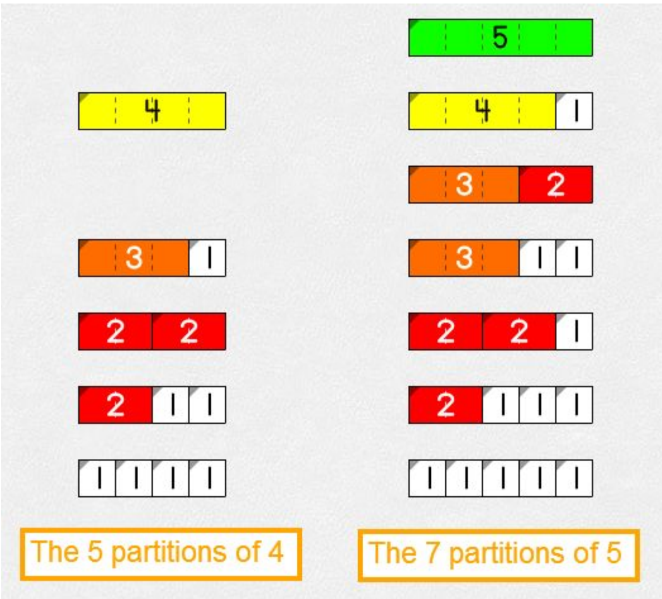


is a justification that

$$\sqrt{20} = 2 \times \sqrt{5}$$

Partitions

If you have been reading all the great Fractions work found on EduGAINS (http://edugains.ca/newsite/math/fractions.html), you will have run into the term "partitioning" and "equi-partitioning". In Number Theory, partitioning refers to the process of breaking a whole number like 5 into whole number addends, like 2 + 2 + 1.



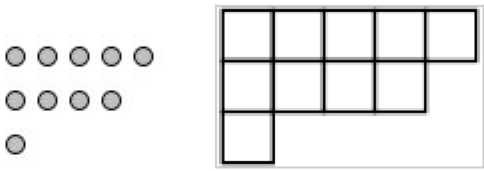
Even the simple diagram above brings out some interesting observations and questions:

- 5 (i.e. the green rod) is considered to be a partition of 5. This is a pretty standard trick in Mathematics that can be somewhat bothersome.
- Order is not considered to be important, so 4 + 1 and 1 + 4 are not different partitions for the purpose of counting. If you do want to consider order to be important, you are talking about *compositions* of the number. Setting the ground rules for counting is crucial.

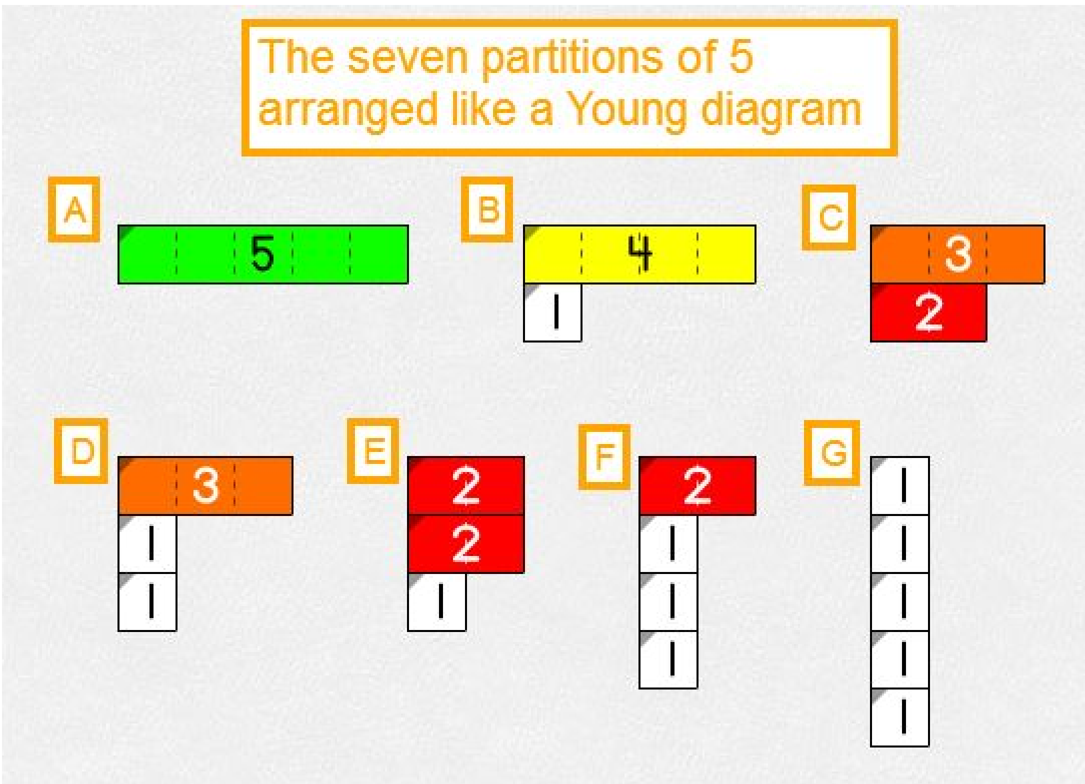
- Organized counting helps make sure you have found all the possibilities. Here, a convention of writing the addends in descending order is used to help organize the count.
- How many more partitions of 5 are there than partitions of 4?
- The diagram above pairs some partitions of 4 with partitions of 5 and not others. What rule is being applied? How might this help count the number of partitions of 6?
- How many partitions of 6 are there? 7? 8? *n*?

If you read the rather technical wikipedia article on partitions, https://en.wikipedia.org/wiki/Partition_(number_theory), you will see two visual representations that mathematicians use to study partitions of whole numbers: the Ferrers diagram and the Young diagram. The Ferrers diagram uses an arrangement of dots whereas a Young diagram uses an arrangement of squares. It is affirming to see how useful visual diagrams are to professional mathematicians studying Number Theory. We should feel confident promoting their use as thinking tools in our classrooms, despite what we might hear from certain pundits.

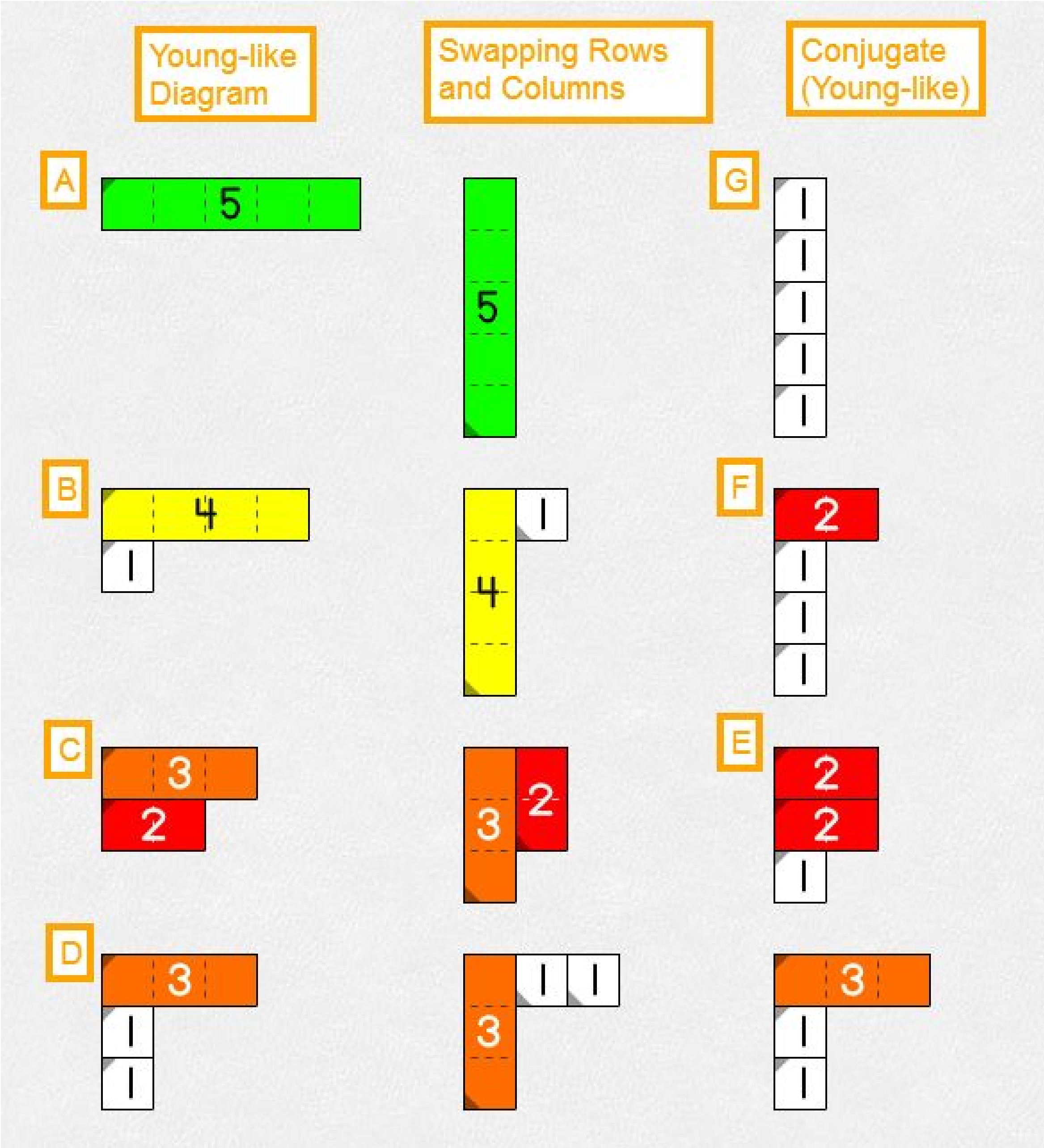
Here, the partition 5 + 4 + 1 of 10 is represented using a Ferrers diagram and a Young Diagram:



The seven partitions of 5 above can be easily rearranged to create Young-like diagrams for each, using relational rods:



It should not surprise anyone that mathematicians like to define operations on mathematical objects, like partitions. One such operation is conjugation (which is a good example of a word that is used multiple ways in mathematics, never mind in French grammar). A conjugate is created by swapping rows and columns in a Young diagram - we might have called it another popular math term: transposition, but were not consulted.



When you swap the rows and columns and turn something like B: 4 + 1 into F: 2 + 1 + 1 + 1, you still have the same number of squares, so each is a partition of the same number. B and F are called conjugate pairs. Students should convince themselves that F's conjugate is B, i.e., that the pairing works "backwards". When the seven partitions of 5 are paired up in conjugates, there must be one left unpaired. How is that possible?

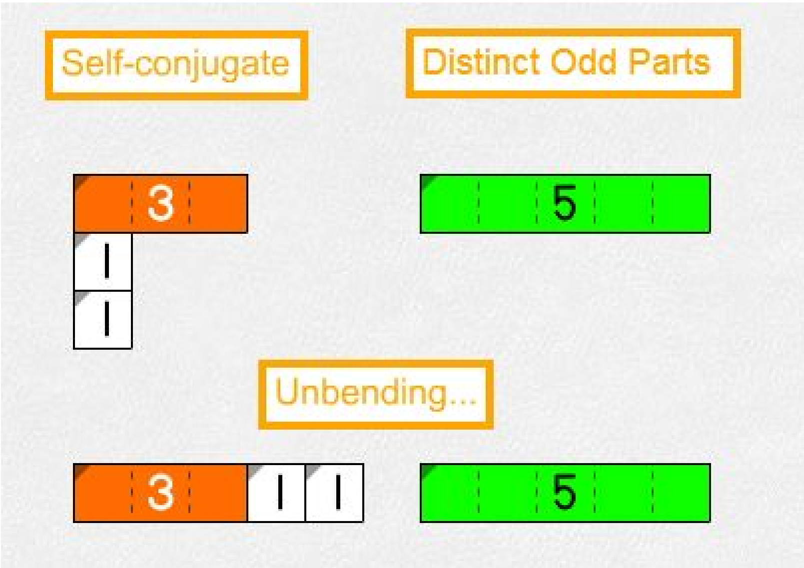
It turns out that D: 3 + 1 + 1 is its own conjugate and so is referred to as a *self-conjugate*. An excellent activity for students would be to come up with lots of other examples of self-conjugates (for bigger numbers) and describe what geometrical properties the Young diagram must have for a partition to be a self-conjugate.

Exercising Spatial Reasoning and Proving

One theorem that affords a visual justification is that:

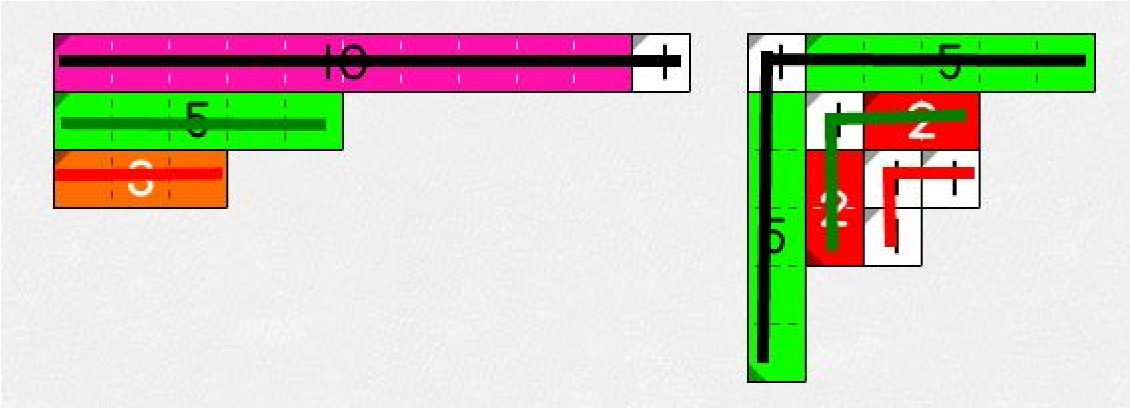
> The number of self-conjugate partitions of a number is the same as the number of partitions with distinct odd parts.

Above, we found one self-conjugate partition, 3 + 1 + 1. There is one partition with distinct odd parts, the bothersome one, 5. Note that 3 + 1 + 1 has only odd parts but they are not distinct since the 1 is repeated. We need a bigger number to get more interesting partitions with distinct odd parts, like the 11 + 5 + 3 used below.



This count of 1 in each case is not particularly convincing for the general result. Students could use their geometric understanding of self-conjugates and a hint about unbending to start playing with lots of other examples.

One of the worst sins in exploration is to provide the spoiler too early and the wikipedia article provides a very nice sketch of the proof that highlights another important counting idea: if you can make a one-to-one correspondence between two sets, they must have the same number of elements (cardinality). Suffice it here to provide a hint of how one partition with distinct odd parts, 11 + 5 + 3, could be bent using Relational Rods to get a self-conjugate.



Notice that 11 is represented by 1 plus 5 in the horizontal direction plus 5 in the vertical direction, which brings us full circle back to how to "see" odd numbers using the tool.

Conclusion

The Relational Rods+ tool for desktop computers and mobile devices provides a platform for rich investigation of number properties using spatial reasoning. The support wiki provides other useful links to resources about Relational Rods, including a YouTube playlist with some other fascinating explorations, including one related to perfect squares.

Feedback and Future Requests

Please feel free to send your feedback about the Relational Rods+ tool using the Feedback Form button inside the Information Dialog. You can also send your comments to WhatsNew@oame.on.ca. You can share worked examples on Twitter using the hashtag #ONmathies. To be among the first to find out about latest digital tool developments, sign up for our email list at http://mathclips.ca/WhatsNewEmailList.html. If you have ideas about nice uses for Relational Rods we would love to receive them and include them on the Support Wiki.