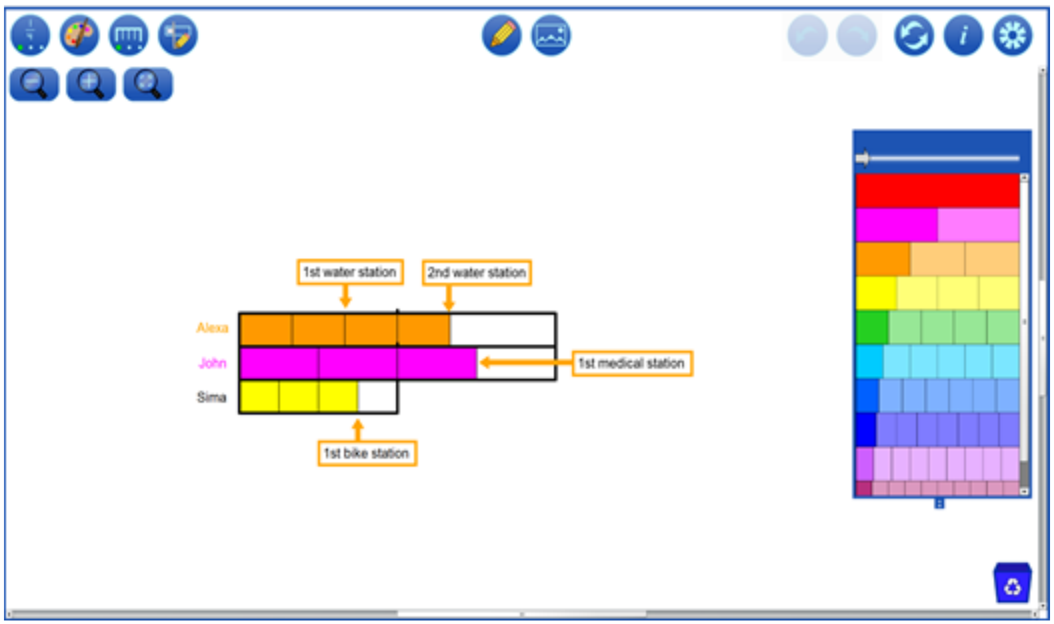
A Fraction Strips Story:

Reflections on a Student's Mathematical Exploration

Contents



A Fraction Strips Story:

Reflections on a Student's Mathematical Exploration

Introduction

This resource was developed to support educators in better understanding how mathies digital learning tools have the potential to support the diverse learning needs of students including those with learning disabilities. Specifically, the Fraction Strips tool will be used to solve a fraction task. In-depth examination of a sample solution will reveal connections between important mathematical ideas related to fractions and key features of the Fraction Strips tool. In addition, explicit connections to cognitive domains and processes are included. The digital learning tools are available from http://mathies.ca.

Why this math content?

Fractions is a content area which can be difficult to learn and teach. In Ontario, we have benefited from ongoing collaborative action research focussed on representing, comparing, ordering and operating on fractions. Highlights from the research are the foundation of the Paying Attention to Fractions [1] document. Connections to this research are made explicit in this document.

> *Spatial visualization has proven to be particularly important for mathematics learning and achievement. Spanning its influence across grades and strands, spatial visualization is a key concept that helps learners both understand and create mathematics. Spatial visualization is a specific type of spatial thinking that involves using our imagination to "generate, retain, retrieve, and transform well-structured visual images".*
>
> (Paying Attention to Spatial Reasoning [2], p 9).

Learning fractions with a focus on spatial visualization enhances instruction and supports students in conceptualizing, reasoning, and thinking about key fraction concepts.

The task and why it was chosen?

The following task was chosen as the basis for discussion:

Bike-a-thon Task:

In a bike-a-thon, cyclists will find

- **water stations every two-thirds of a kilometre**
- **medical stations every three-halves of a kilometre**
- **bike repair stations every three-fourths of a kilometre**

John has reached the first medical station,
Alexa is at the second water station, and
Sima is at the first bike repair station.

Who is furthest along the course? *(Adapted from Nelson Mathematics 6, 2006)*

Representation and comparison of fractions are a focus of the overall and specific expectations in The Ontario Curriculum Grades 1 – 8 Mathematics, and continue to underpin key mathematical concepts in Grades 9-12.

The context of a task can influence the mental images created by students. The Bike-a-thon task suggests the use of a linear model, which research suggests is a powerful model for representing, making sense of, and explaining thinking. Linear models have longevity across grades and mathematical number systems, so it is important to include such contexts within mathematics instruction.

This task was selected to elicit student thinking focused on key fraction concepts, including unit fractions, comparison of fractions to benchmarks, and fractions greater than one whole. Proficiency in comparing fractions is foundational to developing deep understanding of fractions and supports the development of operations involving fractions. This Bike-a-thon task was field tested in a variety of Ontario classrooms which helped to inform the development of this resource.

The Extension Task, “**How much of a lead does John have over Alexa?”**, was included for several reasons:

- The Ontario Curriculum Grades 1 – 8 Mathematics encourages students to pose questions. This extension question is a natural question to pose after answering “Who is furthest along the course?” Educators might ask their students to brainstorm other related questions they might like to explore.
- The solution provided illustrates how the extension question can be answered using a range of strategies. This makes it accessible to students younger than grade 7, when the curriculum introduces the formal subtraction of fractions. Experiences that require students to operate informally on fractions provide solid building blocks on which to eventually learn more formal approaches.
- To appeal to a broader range of educators and grade levels.

Why this mathies tool?

> *Digital technologies allow us to manipulate and see space and spatial relationships like never before. [Such technologies, including many mathies digital learning tools,] allow us to manipulate objects and ideas in ways we could never have done with pencil and paper or chalk and chalkboard. Touchscreen technologies foster gestures and spatial reasoning to build conceptual understanding (Bruce, 2014b). From JK to Grade 12, technology presents opportunities for students to see and even manipulate mathematical ideas in powerful ways.*
>
> (Paying Attention to Spatial Reasoning [2], page 23)

A collection of mathies digital learning tools on mathies.ca has been developed to support the learning and teaching of fractions. These tools have been intentionally designed to incorporate both the fraction research, led by Dr. Cathy D. Bruce, as well as the collaborative action research related to supporting students with learning disabilities in mathematics. Many features of the mathies learning tools have the potential to leverage student strengths and support needs based on diverse cognitive profiles. Fraction Strips was chosen as a tool to highlight since it exemplifies significant connections to the cognitive domains and processes. This story highlights ways that virtual Fraction Strips aid in making precise and accurate representations.

The Fraction Strips tool provides the opportunity for students to explore, investigate and, with the support of teachers, connect conceptual understanding with procedural fluency. As students use this tool to solve a variety of problems, they make connections between and among fraction concepts, enabling them to think deeply and apply these concepts and skills in novel situations. The visual representations created by the Fraction Strips tool make explicit the mathematical structure of important fractions concepts, such as part-whole relationships and unit fractions.

Structure of the Document

The information in this document is presented in table format. Each column deals with a specific aspect of the discussion while each row examines one step in the sample solution in detail.

Column 1: Solving the Problem - This column provides a step by step sample solution of the bike-a-thon task using the Fraction Strips tool. Used in a number of classrooms, students have generated a variety of different solutions using various tools and strategies. The solution presented in this document, attributed to a fictitious student named Riley, was designed to highlight specific mathematical ideas and tool features.

Column 2: Mathematics Made Explicit - This column identifies the important fraction concepts, identified through research, that are foundational and result in powerful learning experiences for students. These fraction concepts are explicitly identified and explained in the context of each step of the sample solution. The Paying Attention to Fractions [1] document informs this column and references to it are abbreviated as *PATF*.

Column 3: Cognitive Domains/Processes Made Explicit - This column identifies how the use of the Fraction Strips tool leverages student strengths and supports student needs based on the cognitive domains and processes [3]. These considerations assist with program planning and apply to all students, including those with identified learning disabilities.

Ways to Use this Resource

Educators may consider using this resource during whole school learning, such as staff meetings, lunch-and-learns, school-based professional learning communities, and/or board-sponsored collaborative inquiries.

Educators, including classroom teachers, special education teachers, school leaders, school math leads, math coaches, math and special education professional learning facilitators are encouraged to use this resource to support their math learning goals and student learning needs.

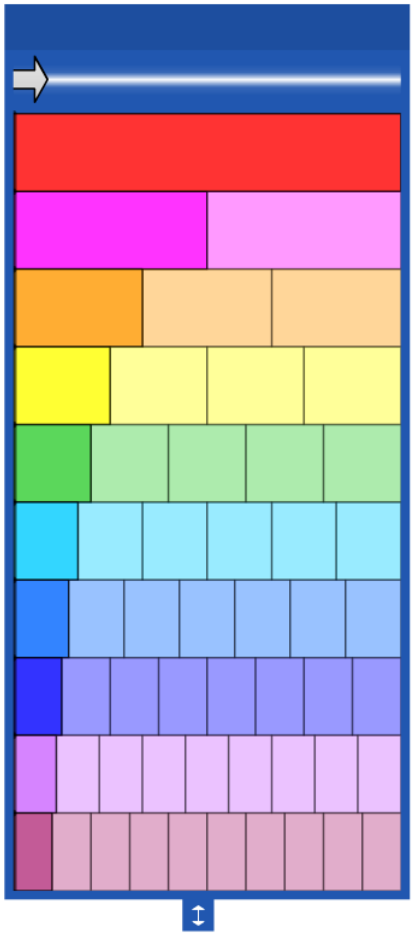
Before using this resource, it is recommended that educators generate a variety of possible solutions for the Bike-a-thon task, including some using the mathies Fraction Strips tool.

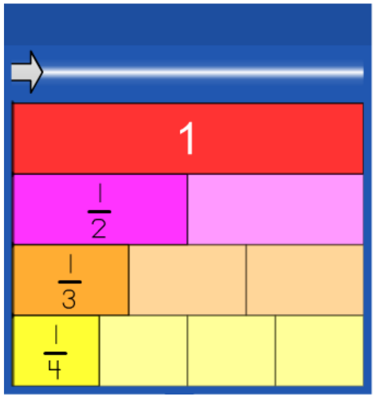
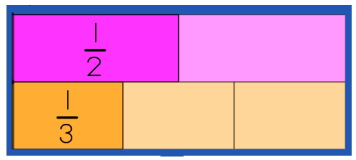
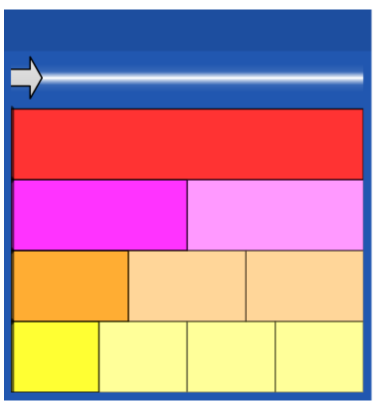
It is anticipated that this resource will be used by educators in diverse ways. Below are some suggestions for learning with this resource.

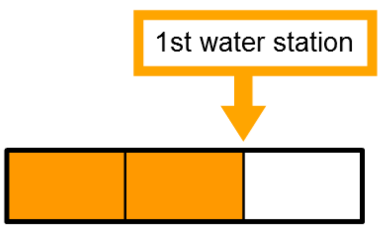
- Compare and contrast your solutions to the one presented in this resource. Make connections to the content within the 'Making Math Explicit' and/or 'Making Cognitive Domains/Processes Explicit' columns
- Anticipate potential incorrect solutions or misconceptions that students might have and consider how to address them
- Examine "Making Math Explicit" column to intentionally plan for instruction and assessment (e.g., make explicit the mathematical structures revealed through Fraction Strips representations, develop conceptual understanding of key fraction concepts and make connections to facts and procedures, respond to student thinking, address misconceptions)
- Explore the potential of mathies digital learning tools, such as Fraction Strips, to leverage student strengths and support student needs; consider how these digital tools can serve as assistive technology for students with learning disabilities
- Explore how using the Fraction Strips tool as the site of the problem solving connects to other mathematical processes (representing, reasoning and proving, communication etc.)
- Explore the role of mathies digital tools, such as Fraction Strips, as a way to enhance student math thinking and learning with concrete manipulatives
- Explore connections to the Paying Attention to Fractions [1] and Paying Attention to Spatial Reasoning [2] documents.

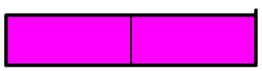
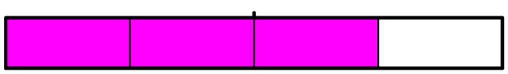
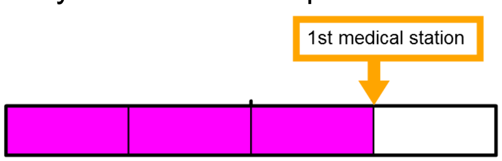
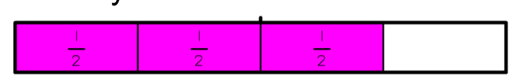
By examining one specific task deeply, educators will have the opportunity to see the Mathematics revealed in new ways and consider pedagogical approaches to address diverse student learning needs.

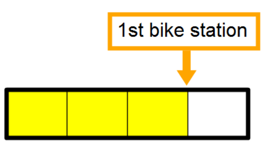
The Bike-a-thon Task

Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
Riley opens the Fraction Strips tool and considers which fractions apply to the problem. 	Notice that the tower includes fraction strips that are equi-partitioned into pieces. (PATF Partition, p. 6) The Fraction Strips tool does not show labels on the pieces by default. As students reason about the part-whole relationship represented in the tower, they count the number of pieces that make up a whole and name them as fractions, e.g., there are five equal-sized green pieces that make up a whole, so one green piece represents $\frac{1}{5}$. (PATF Part-Whole Relationships, p. 5)	**Executive Functioning** Using the Fraction Strips tool supports organization of materials and provides the workspace to represent thinking. **Perceptual Reasoning** The fraction strips tower supports understanding of part-whole relationships represented visual-spatially. A common whole is retained for all fractions, allowing for accurate interpretations of the relationships within part-whole fractions as well as the meaning of the numerator and of the denominator. The size of the individual pieces helps students to understand fractions as quantity (e.g., students see that $\frac{1}{10}$, which is a small piece of the whole, is closer to zero than $\frac{1}{4}$). **Visual-Motor Integration** Use of Fraction Strips ensures an accurate representation instead of drawing by hand. **Memory** Associating a colour with a fractional quantity may further support some students with differentiation of fractions.

Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
Optional Step: Some students might turn on the fraction labels  . 	The unit fractions in the tower are ordered from largest to smallest, addressing the common misconception that unit fractions with larger denominators are larger. (PATF Comparing and Ordering, p.18) Here, the fraction labelled $\frac{1}{3}$ is clearly smaller than the one labelled $\frac{1}{2}$. 	**Perceptual Reasoning** Pairing of visual and numeric representations by turning on the fraction labels further supports the understanding of both part-whole relationships and fractions as quantity.
Optional Step: Some students might customize the tower using the settings to show only the fraction strips related to the task. 	Reducing the number of strips might help a student to focus on the relevant fractions. The advantage of showing all strips is in constantly seeing the relationship between successive unit fractions.	**Processing Speed** Fewer fraction strips will minimize time and fatigue to process visual information.

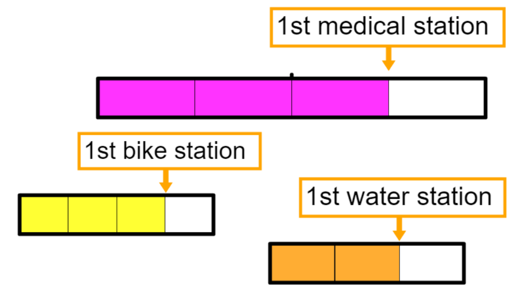
Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
Since cyclists will find water stations every two-thirds of a kilometre, Riley looks for a piece to represent thirds. Riley recognizes the orange colour as representing one-third  and drags 2 pieces out to represent 2 one-thirds, the position of the first water station.  Riley then labels the representation using the Annotation tools .  	Notice that when the orange piece is dropped into the workspace, the whole is explicitly shown. Two-thirds means a count of 2 fractional units of one-third. Its representation shows $\frac{2}{3}$ of a whole coloured orange. It also shows $\frac{1}{3}$ of a whole not coloured. (PATF The Whole, p. 14) Fraction pieces snap together ensuring a continuous representation of the fraction (no gaps). This ensures an accurate representation is created which is referred to as a *train*. By iterating (i.e., making a copy of) the same unit fraction, a new fraction with the same fractional unit (denominator) and a different count (numerator) is created. (PATF Iterating, p. 12) Fraction pieces are a rectangular area model. Their value is based on their relationship to the *area of the whole*. However, since they all have the same height, their value can be found by comparing the length of the piece to the *length of the whole*. (PATF Area Model, p. 4)	**Perceptual Reasoning** The Fraction Strips tool allows a student to create an accurate visual model which incorporates a representation of the whole and can be easily manipulated, annotated and revised without losing precision. **Working Memory** The tool reduces the cognitive load on a student who must simultaneously track the count (numerator), fractional unit (denominator) and the whole by displaying all visually. Regular annotation during problem solving with the tool can also assist students with working memory. **Fine Motor** Fraction pieces snap together. This feature alleviates student frustration and enables a focus on mathematical thinking rather than struggling with the alignment of pieces. **Fine Motor/Processing Speed** A student can zoom in  on all the pieces to make them bigger which supports fine motor control by making it easier to drag pieces into and around the workspace. Resizing the pieces can also support students in processing visual-spatial information represented in the tower.

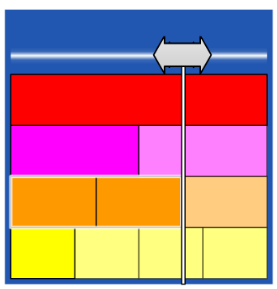
Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
Since there are medical stations every three-halves of a kilometre, Riley looks for a piece to represent halves. Recognizing that the pink colour represents one-half, Riley drags three pink pieces out, counting 1 one-half,  2 one-halves,  3 one-halves,  to represent the position of the first medical station. Riley then labels that position.  	Any completely coloured whole is indicated by a black tick at the top right. Dragging unit fractions out to create a model, in this case of $1\frac{1}{2}$, engages students in using unit fractions to compose another fraction. Students can connect the composition to unit fractions counting *'1 one-half, 2 one-halves (or 1 whole), 3 one-halves.'* This type of composition builds fraction number sense. (Math for Teaching: Developing Proficiency with Partitioning, Iterating & Disembedding [4]) Students see the equivalence of *3 one-halves* and $1\frac{1}{2}$. This train shows two wholes, the first is completely coloured and the second is partially coloured. (PATF Types of Fractions, p. 11) Reading a fraction for its quantity (e.g., $\frac{3}{2}$ as "3 one-halves"), rather than a digit read, (e.g., "3 over 2") helps establish the fraction as a single number rather than two independent numbers. It indicates the unit fraction (one-half) and the count of these units (3) which is the total quantity, in this case, the distance from the start of the bike-a-thon. (PATF Unit Fractions, p. 12)	**Perceptual Reasoning** Naming (aloud) a fraction for its quantity, with the support of a visual representation, enhances students' understanding of part-whole relationships. **Memory / Verbal Comprehension** Pairing of Fraction Strip representations with counting aloud may • provide multiple sources of input (i.e., visual-spatial, auditory, written), • ease the load on working memory, and • facilitate conceptual understanding which in turn supports long-term memory. **Memory** A student might use the settings to show the labels to further ease the cognitive load on working memory and support long-term memory.  **Executive Functioning** This provides an opportunity for self-monitoring. Students can reflect on the connections between counting aloud by the unit fraction, the numeric representation for the count, and the visual representation of the fraction.

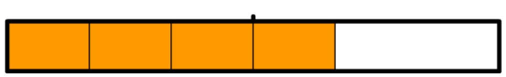
Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
Since bike repair stations are found every three-fourths of a kilometre, Riley looks for a piece to represent one-fourth. Recognizing that the yellow fraction strip is partitioned into four equal pieces, Riley drags one yellow one-fourth piece to the workspace.  Riley presses the copy button  to make two more one-fourth pieces so that there are 3 one-fourths altogether.  Riley notes that Sima's position is only one-fourth away from a whole. As before, Riley labels the first repair station. 	The student composes $\frac{3}{4}$ by iterating 3 one-fourths. The '3' is the count of the fractional unit, one-fourth. (PATF Operations, p. 19)	**Perceptual Reasoning** Some students may notice both the fraction and its complement, and when combined, a whole is created. The visible whole supports paying attention to this important part-whole concept. (PATF Part-Whole Relationships, p. 5) **Fine Motor** The annotation text tool allows students to use the keyboard to type instead of the pencil tool to hand-write annotations. On mobile devices, the built-in speech-to-text tool can be used. The results of dictating a fraction may vary based on device; you may get three-fourths, ¾ or 34.

<table>
<tr><th>Solving the Problem</th><th>Mathematics Made Explicit</th><th>Cognitive Domains/Processes Made Explicit</th></tr>
<tr><td colspan="3">At this point, Riley has represented the position of each of the first stations.

Riley now re-reads the task and thinks about where each person is.

John is at the first medical station,
Sima is at the first bike repair station,
but Alexa is at the second water station.

</td></tr>
<tr><td>Since Alexa is at the second water station, Riley recognizes that another water station needs to be added to the representation.

Riley drags the equivalence bar to highlight 2 of the one-third pieces, creating $\frac{2}{3}$.


Riley drags $\frac{2}{3}$ out and connects these pieces to the 2 one-third pieces that are already in the workspace, so that she has 4 one-thirds.
</td><td>Notice that the pieces are highlighted as you move the equivalence bar along the length of each strip. Students will notice fractions that are equivalent to $\frac{2}{3}$ since they are highlighted. Students may also recognize that $\frac{2}{3}$ and $\frac{1}{3}$ add to a whole.

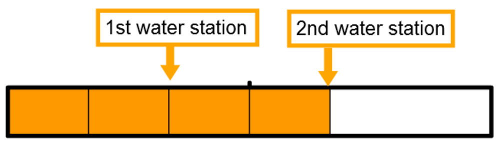
When the student puts $\frac{2}{3}$ and $\frac{2}{3}$ together in the model, they are essentially adding the two fractions. This is an introduction to the notion that adding fractions involves combining the counts (numerators) while retaining the fractional unit (denominator).

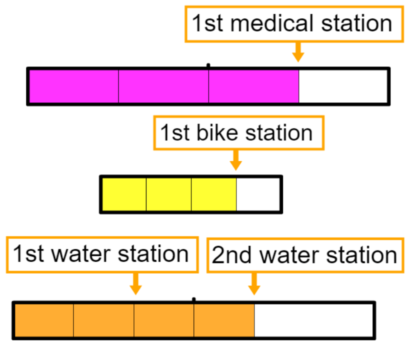
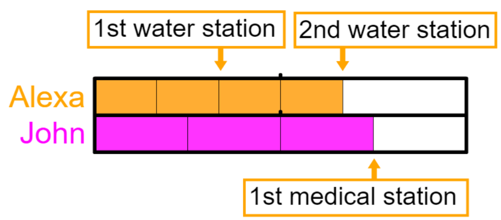
Similarly, note that if a student considered this as doubling $\frac{2}{3}$, they may think of that as doubling the count (numerator) while retaining the fractional unit (denominator). Throughout the solution to this problem, Riley has many such opportunities to informally explore operations. (PATF Operations, p. 23)</td><td>Perceptual Reasoning
The equivalence bar helps students visualize the two-thirds as a new unit that is made of 2 one-thirds. This new unit can be used for representing and operating with other fractions. (e.g. to visually combine two-thirds and two-thirds to make four-thirds)

Working Memory
The creation of $\frac{2}{3}$ as a unit supports students in holding and manipulating the quantity.

Fine Motor
Using the equivalence bar, as well as dragging out a set of 3 pieces is easier than having to drag out 3 single pieces, one at a time.

Verbal Comprehension
Paying attention to actions with the fraction pieces provides a powerful form of nonverbal communication (e.g. using the equivalence bar to create $\frac{2}{3}$ vs dragging 2 one-third pieces), making mathematical thinking visible.</td></tr>
</table>

Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
Riley notices that the representation of 4 one-thirds also shows one whole and one third. This is a visual confirmation that $\frac{4}{3} = 1\frac{1}{3}$. Riley then labels the second water station position.  	Alternatively, the $\frac{2}{3}$ can be created by clicking on the second one-third piece in the tower and dragging both pieces to the workspace. This action suggests that $\frac{2}{3}$ is being thought of as a unit. (PATF Let's Recap Part-Whole Relationships, p. 6) The student may count the unit fraction pieces and see $\frac{4}{3}$ or count the whole and the piece beyond the whole to see $1\frac{1}{3}$. In this way, students gain both a conceptual understanding and flexibility in naming equivalent fractions greater than one whole as either mixed or improper fractions. (PATF Types of Fractions, p. 11)	**Perceptual Reasoning** The Fraction Strips representation for fractions greater than 1 supports students in visualizing fractions in two ways, through the count of the unit (4 one-thirds) or the whole with remaining units (1 whole and one-third), supporting flexibility and fluid reasoning. **Long-Term Memory** Developing understanding of a concept (such as the relationship between improper and mixed fractions) through visual/digital representations creates memorable experiences and supports a transition from short-term to long-term memory.

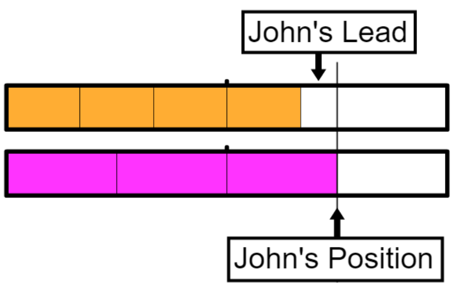
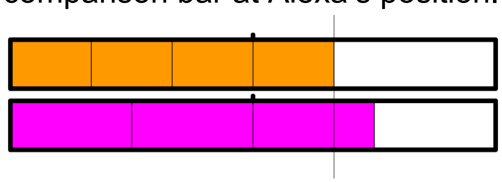
Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
 Riley examines the three trains and notices that the $\frac{3}{4}$ is less than one so then focuses on comparing the two remaining fractions.	Benchmark fractions can be used to compare these distances: $\frac{3}{4}$ is less than 1, and $\frac{4}{3}$ and $\frac{3}{2}$ are both greater than 1. (PATF Comparing and Ordering Using Benchmarks, p. 18)	**Executive Functioning** Fraction Strip representations focus attention on key mathematical concepts (e.g., fraction as quantity, complement), preserving the cognitive demand of the task, i.e., comparison of fractions, and providing opportunities for self-monitoring.
Riley moves the remaining trains so they have the same starting position and then adds the names of each biker to the start of each train.  Riley compares Alexa's position of $1\frac{1}{3}$ to John's position of $1\frac{1}{2}$. Since the unit fraction $\frac{1}{2}$ is greater than the unit fraction $\frac{1}{3}$, $1\frac{1}{2}$ is greater than 1 ⅓. This means that John is in the lead.	Aligning the trains on the left can help to visually compare their values. Here, the left edges of the trains are snapped to a position that represents the starting line for the bike-a-thon so that the end positions can be compared. Some students might compare the white space in each model – seeing the space to the right of the orange $\frac{1}{3}$ as being larger than the space to the right of the last pink $\frac{1}{2}$. So $1\frac{1}{2}$ is greater than $1\frac{1}{3}$ since there is less white space to get to two wholes.	**Perceptual Reasoning** Visual/digital representations provide an opportunity for visual-spatial reasoning and support the development of visualization (e.g., comparing fractions with a focus on quantity, benchmarks). The snap-to-grid feature enables an accurate comparison of the fractions, making explicit key concepts embedded within the comparison (e.g., common wholes, fraction as quantity) **Fine Motor** The snap-to-grid feature minimizes the challenge of alignment of trains and pieces for comparison.

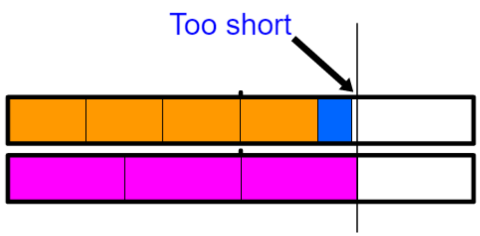
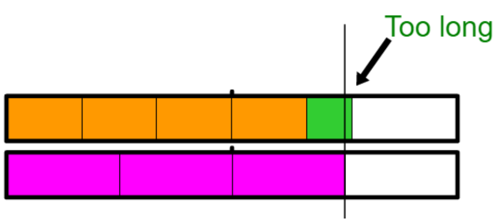
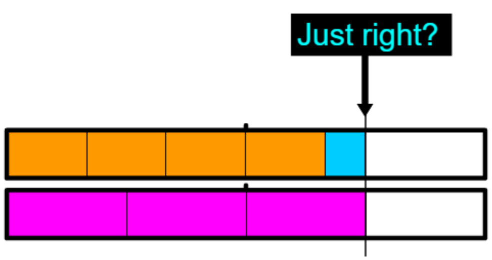
An Extension Task

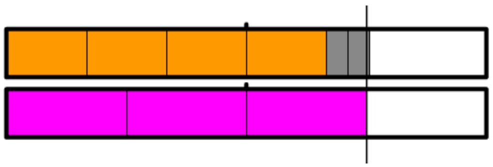
Students should be encouraged to think about the questions that are raised after solving a problem. One such problem might be:
How much of a lead does John have over Alexa?

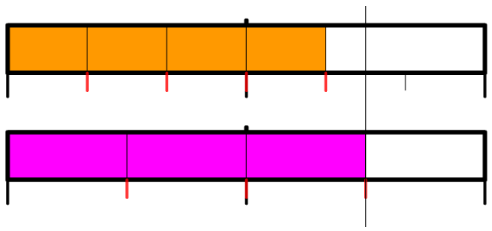
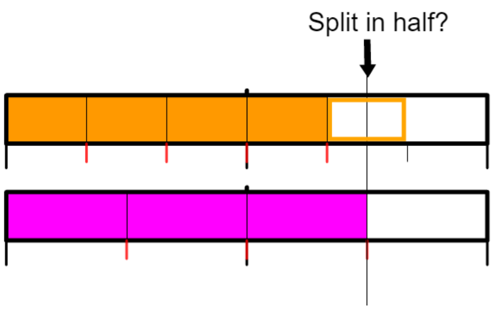
This extension problem creates an opportunity for the informal exploration of operations involving fractions using visual representations within a problem-solving context. Using visual representations to support students in their reasoning and proving skills allows for a rich exploration of this type of problem that relies on the students' number sense rather than on their facility with an algorithm, such as finding common denominators. (PATF Comparing and Ordering Constructing Models[1], p. 18) If a student recognizes this as a subtraction of fractions problem and has been taught to calculate a common denominator they may choose to proceed symbolically. (PATF Constructing Models[1], p. 18)

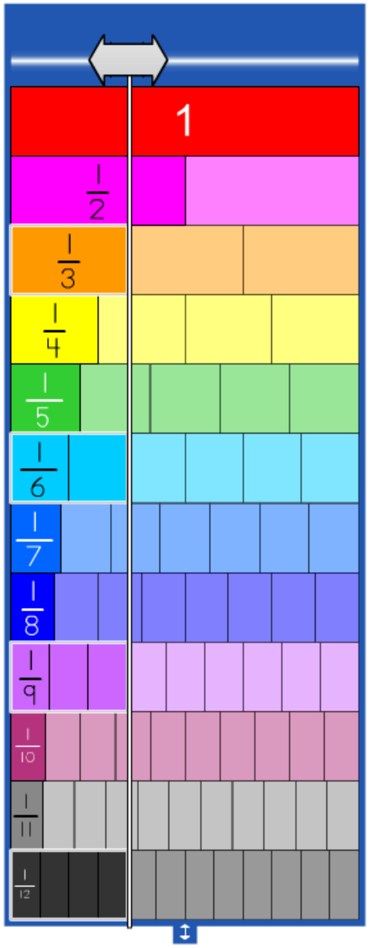
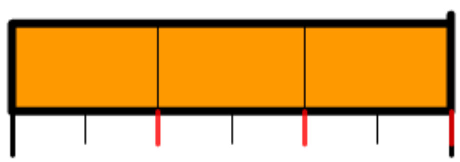
When quantifying the lead, we are comparing the two positions. Comparison is one of the situations in which you use subtraction (Problems Involving Addition and Subtraction [5]).

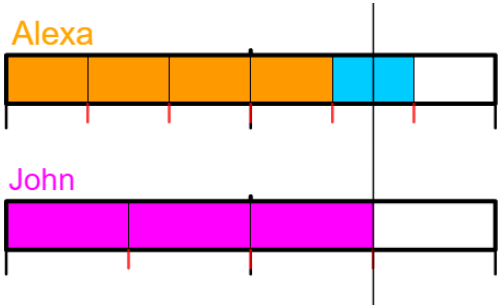
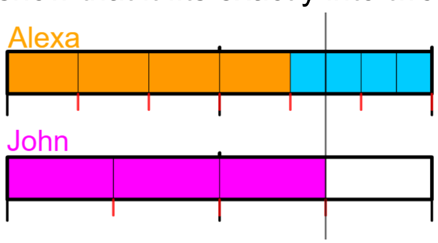
Solving the Problem	**Mathematics Made Explicit**	**Cognitive Domains/Processes Made Explicit**
Riley adds a vertical comparison bar and drags it to John's position.  Riley realizes that John's lead is the size of the chunk from the end of the orange pieces to the comparison bar and decides to find this difference.	It is interesting to think about how the solution might be different if Riley put the comparison bar at Alexa's position.  Riley could have focussed on taking away the orange quantity from the pink quantity to the find the difference, rather than thinking about the amount to add on.	**Perceptual Reasoning** The comparison bar supports visualization of the difference (John's lead) as a quantity. Visual-spatial reasoning involving the negative space (missing quantity) or extra quantity can support students in selecting a strategy that models their thinking (non-verbal reasoning).

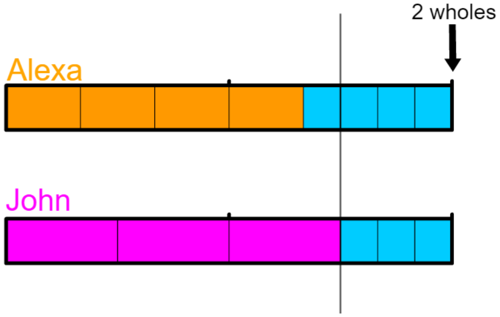
Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
Riley looks at the tower to find a piece that looks to be about the right size. Riley drags out $\frac{1}{7}$ but it is too short.  Riley drags out $\frac{1}{5}$ but it is too long.  Riley drags out $\frac{1}{6}$ and it looks just right. 	The sample solution shown here uses Trial and Error as a problem solving strategy. The student begins by estimating the quantity that makes up the difference. A student who drags out unit fractions might notice that unit fractions with a larger fractional unit (denominator) are smaller than ones with a smaller fractional unit. For example, $\frac{1}{5}$ is bigger than $\frac{1}{6}$, even though 5 is less than 6. For some students, this is counter-intuitive, especially if they are trying to relate the denominator of a fraction to a whole number understanding of comparison. A visual representation can quickly dispel this misconception. Students will use their visual-spatial reasoning skills (Paying Attention to Spatial Reasoning [2]) to hold an image of the empty white space in their mind while looking for a piece of the same size in the fraction tower. This is an example of disembedding. (Math Teaching for Learning:Developing Proficiency with Partitioning, Iterating & Disembedding [4])	**Perceptual Reasoning** Reasoning and proving by testing different fraction pieces supports students in visual-spatial reasoning, focusing attention on fraction as quantity, and reinforcing connections between visual and numeric representations of fractions.

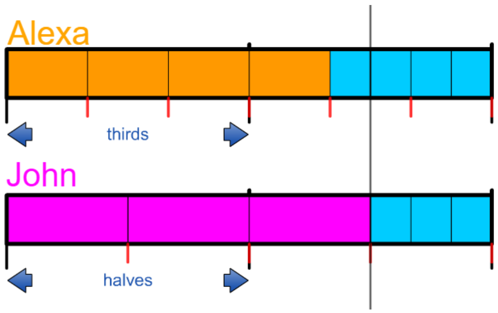
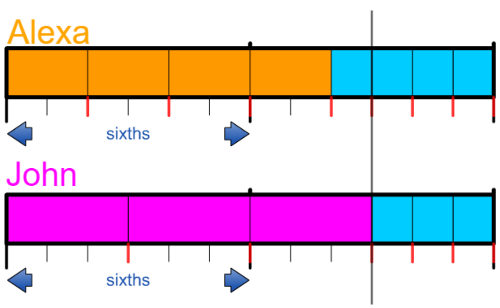
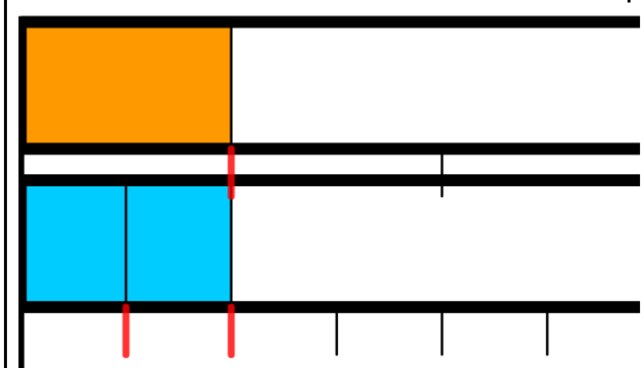
Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
Riley is not sure whether $\frac{1}{6}$ is exactly right since $\frac{2}{11}$ looks just about as good.  *Note:* Riley customizes the tower making elevenths visible. *Optional Step:* Riley could Zoom In to realize that $\frac{2}{11}$ does not look to be as close to the right size as it did originally.	*One distinction from whole numbers is the density of fractions, in that it is always possible to identify a fraction that lies between two other fractions on the number line (e.g., the fraction $\frac{5}{14}$ lies between $\frac{2}{7}$ and $\frac{3}{7}$). This is not true for whole numbers, as there is no whole number between one and two.* (Foundations to Learning and Teaching Fractions: Addition and Subtraction [7], p. 34) Because of this, the use of the vertical comparison line is not sufficient to reason about quantities that are close in value. Zooming In makes it possible to make more accurate conclusions. The Fraction Strips tool has a handy Zoom to Fit button which ensures that all pieces and the tower are visible. Using the Zoom Out or Undo button is useful to restore a previous view.	**Processing Speed** Zooming in can support visual-spatial processing of fractions that are relatively close in value.

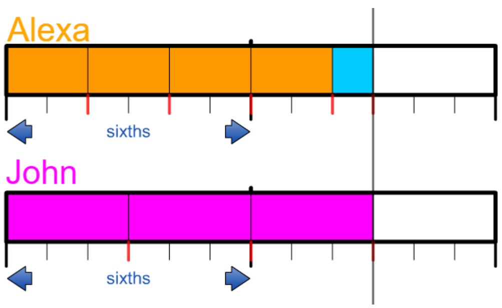
Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
Riley turns on the rulers  so that tick marks appear.  Riley thinks about the relationship between John's lead and the next one-third in Alexa's orange train, indicated below.  It looks like the comparison bar splits that one-third in two equal pieces. This makes Riley think that John's lead is $\frac{1}{2}$ of $\frac{1}{3}$.	The tick marks are illuminated in red when they are perfectly aligned to the end of a piece, enabling precise reasoning. The ruler supports students in paying attention to the length of the piece rather than its area. As such, the Fraction Strips tool supports students as they transition to representing fractions as lengths or positions on a number line. The student is able to visualize that the needed piece is half of $\frac{1}{3}$, leveraging spatial reasoning. (Paying Attention to Spatial Reasoning [2], p. 9)	**Executive Functioning** The illuminated red ticks draw attention to the unit fraction in each representation (i.e., one-third and one-half) and with the comparison bar prompts thinking about how to determine John's lead. **Perceptual Reasoning** Use of the comparison bar supports visualization of John's lead as a quantity, i.e. the difference between John's position and Alexa's position, and prompts visualization of the relationship between John's lead and the unit fraction one-third. **Visual Processing** The comparison bar provides a visual anchor of John's lead (quantity), enabling a focus on reasoning about the difference.

Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
Riley uses the equivalence bar in the fraction tower to look for fractions equivalent to $\frac{1}{3}$.  Riley notices that 2 one-sixths is equivalent to $\frac{1}{3}$, which means that $\frac{1}{6}$ is half of $\frac{1}{3}$.	The equivalence bar provides equivalent ways of naming $\frac{1}{3}$ (as $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$ etc.) and allows you to see that one-half of $\frac{1}{3}$ is $\frac{1}{6}$ or $\frac{2}{12}$. Dividing a number into 2 equal groups can be thought of as finding $\frac{1}{2}$ of the number which is the same as multiplying the number by $\frac{1}{2}$. (Math for Teaching: Ways We Use Fractions, Fraction as Operator[8]) Students might reason that if each of the three one-thirds that make up a whole is split in half, then there would be six equal pieces in the whole, each measuring one-sixth of the whole. (PATF Unit Fractions, p. 12) 	**Perceptual Reasoning** Using the equivalence bar enables visualization of equivalent fractions either through equi-partitioning or merging of equal parts. This supports students in understanding that a quantity can be named with different numeric representations (e.g., $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$). **Memory** Highlighting equivalent fractions using the equivalence bar supports conceptual understanding and provides a memory trigger for generating equivalent fractions. **Executive Functioning** The equivalence bar and the ruler with illuminations focus attention on the structure of the fraction representations. **Perceptual Reasoning** The equivalence bar and the ruler draw attention to the relationships between the representations of equivalent fractions.

Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
Riley places two one-sixth pieces to make the next one-third in Alexa's orange train.  So, the difference between John and Alexa's positions does seem to be one one-sixth.	Although, Riley has reasoned that one one-third is equivalent to two one-sixths, it has not yet been proved that John's position is exactly at the end of the first one-sixth piece. Many students might have thought they were done at this point. Sometimes it is very hard to decide what you have to justify and what is "obvious" in a proof. Here, the idea that the comparison bar splits the next one-third in Alexa's orange train is a conjecture that requires justification.	**Perceptual Reasoning** Creating and manipulating these representations enables and supports non-verbal reasoning (e.g., using visual models to make and test conjectures). **Executive Functioning** The dynamic nature of the Fraction Strips tool supports making and testing conjectures and provides immediate feedback for students, thus supporting self-monitoring and establishing strategic next steps.
Riley sets out to prove that the difference is exactly one-sixth. After filling in the remaining empty space in Alexa's orange train with 2 one-sixth pieces, Riley notices that the last tick illuminates to show that it fits exactly into two wholes. 	To be able to establish equality, a benchmark is needed. Here, Riley is essentially working backwards from the benchmark 2.	**Working Memory** The visual representations created with Fraction Strips support students in holding multiple quantities simultaneously as they are visible, enabling a focus on processing and manipulating the quantities to reason and prove.

Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
Riley then fills in the empty space in John's pink train with 3 one-sixth pieces and notices that the last tick illuminates to show that it fits exactly into two wholes.  John's position is 3 one-sixths less than two wholes. Alexa's position is 4 one-sixths less than two wholes. This convinces Riley that John's lead is exactly 1 one-sixth.	Since Alexa's position plus the first one-sixth ends at a position three one-sixths from 2 and John's position also ends at a position three one-sixths from 2, John's position is one one-sixth bigger than Alexa's. This image is nicely set up for students to notice equivalences like $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$ (informal operations).	**Perceptual Reasoning** The iteration of one-sixth to complete the second whole is a way to prove the difference of one-sixth. The tool lends itself to multiple access points for proving; either with the original fractions represented ($1\frac{1}{2}$ and $1\frac{1}{3}$), or the complement ($\frac{1}{2}$ and $\frac{2}{3}$, the white space at the end of each whole).

Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
Riley wonders whether one-sixths can be used to name all the relevant lengths. Riley clicks the ruler button  one more time to also show the steppers.  Riley clicks the up-arrows to change both steppers to sixths and makes note of the red illumination. 	Using the ruler allows the student to think about the pieces being named using other fractional units (e.g., each one-third piece is split into two to make two one-sixths, doubling the number of pieces) (PATF Equivalency, p. 14) A student might have displayed the rulers much earlier and adjusted the steppers to find a way of naming both fractions using the same units. If they did this using trial and error, they might have missed an opportunity for deeper thinking about how one-sixth is related to one-third and one-half.	**Perceptual Reasoning** The visual tool aids students in understanding that a fraction can be named in an infinite number of ways as they use the stepper to change the fractional unit used. Experimenting with different fractional units allows the student to review and revise their thinking. **Processing Speed** Zooming to a particular portion of the visual representations can support students in reasoning about the quantities and the ruler’s red illuminated ticks. 
Riley notices that 4 one-thirds can be	By splitting unit fractions to create	**Working Memory**

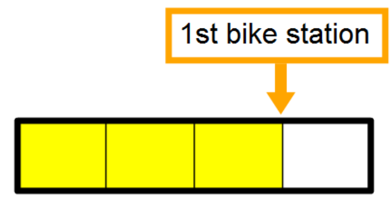
Solving the Problem	Mathematics Made Explicit	Cognitive Domains/Processes Made Explicit
named as 8 one-sixths. Similarly, 3 one-halves can be named as 9 one-sixths. The difference between 9 one-sixths and 8 one-sixths is 1 one-sixth. Riley goes to settings  and saves the file . "Cool!", says Riley, finally satisfied. 	common fractional units, students are able to make connections to the concept of common denominators (common fractional units) and how they are used in addition/subtraction of fractions. $\frac{3}{2} - \frac{4}{3}$ $= \frac{3 \times 3}{2 \times 3} - \frac{4 \times 2}{3 \times 2}$ $= \frac{9}{6} - \frac{8}{6}$ $= \frac{1}{6}$ (PATF Equivalence and Comparing, p. 23) However, moving students too quickly to this idea of common denominator, might interfere with student thinking, problem solving, and reasoning.	The ruler with illumination supports students to hold the original fraction quantities (i.e., $\frac{4}{3}$ and $\frac{3}{3}$) and rename them by changing the partitions, enabling the visualization of $\frac{8}{6}$ and $\frac{9}{6}$.

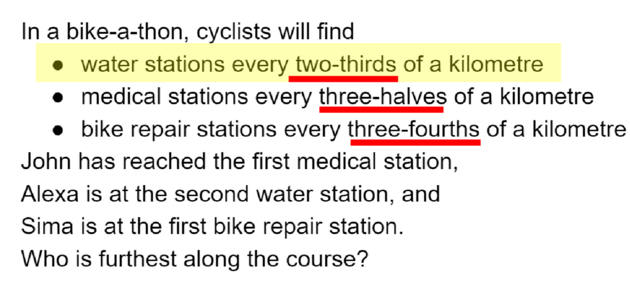
Common Features of mathies Learning Tools

How do they support student learning?

Common Feature	Teaching Considerations	Cognitive Domains/Processes Made Explicit
Interactivity The elements provided by the tools are quick to create and easily moved, duplicated and manipulated.	When students do not need to worry about making accurate representations themselves, they can concentrate on the mathematical aspects of their work. Many students find working with virtual manipulatives more engaging than working with a limited set of physical manipulatives. Often students will persevere longer when using digital tools, trying different approaches as compared to when using pencil and paper. Depending on the materials available in the classroom, virtual manipulatives may be easier for a teacher to access and manage than physical manipulatives. Students benefit from access to an endless supply of configurable representations when they use virtual manipulatives. They benefit from more direct and kinesthetic interactions when using physical manipulatives.	**Perceptual Reasoning** Thinking, reasoning and conceptualizing are promoted through dynamic interactions with the mathies digital learning tools. **Visual-Motor Integration** Creating representations with mathies digital learning tools as opposed to by hand minimizes inaccuracy and frustration. **Executive Functioning** A digital environment provides the necessary tools to represent and communicate thinking without requiring the organization and management of a variety physical materials. The dynamic nature of mathies digital learning tools supports making and testing conjectures, providing immediate feedback for students. This promotes self-monitoring and strategic decision making. **Memory** Use of mathies digital learning tools supports the development of conceptual understanding, enabling storage in and retrieval from long-term memory.

Common Feature	Teaching Considerations	Cognitive Domains/Processes Made Explicit
Settings Tools allow for various customizations by changing settings either from an opening dialog, workspace button or settings dialog. Changing a setting allows the student to exercise personal choice or make adjustments to meet a perceptual or other learning need.	Setting up the elements in the workspace of a mathies digital tool may involve some important instructional decisions. For example, the Fraction Strips tool and the Relational Rods tool allows the objects in the workspace to either show a numeric label or not. Showing the labels might reduce the cognitive load on a student but also might limit the sense-making needed for their conceptual development. Generally, labels might be more helpful once a student has had the introductory experiences necessary to ground their thinking.	**Processing Speed** Being able to resize or zoom in will help students focus on particular aspects of the representation.
Undo/Redo Undo is useful for correcting a faulty or unexpected action. Undo can also be used to return the tool to the start of a sequence of steps. Redo can then be used to review those steps or explain them to someone else.	**Observational Assessment** Being able to undo/redo encourages risk taking. A teacher can better assess student thinking by reviewing all the steps taken using a tool rather than by looking at a final static image.	**Executive Functioning** Undoing and redoing provides opportunities for students to engage in self-assessment. This may provoke refinements to their solution. **Memory** Undoing and redoing provides visual prompts which can trigger a student's memory of their math thinking. Pairing student think-alouds with this process can further support memory. **Perceptual Reasoning** Paying attention to student actions with tools can provide significant understanding of their math thinking, even if not accompanied by verbal or written language.

Common Feature	Teaching Considerations	Cognitive Domains/Processes Made Explicit
Annotation Freehand drawing, lines, shapes and text boxes can be added to the workspace.	The mathies digital learning tools are effective sites of problem solving for students. The annotation tools give students opportunities to support their reasoning by adding notes, explanations and wonderings. The annotation tools give students opportunities to communicate their thinking and support their conclusions.	**Working Memory** Using annotations as students work through a solution enables tracking of key information. *Example:*  **Visual-Motor Integration** Choosing the annotation text tool in lieu of the pencil tool can alleviate the challenge of visual-motor integration. When used on mobile devices, the built-in speech tool can be used in the text tool mode which provides additional support.

Common Feature	Teaching Considerations	Cognitive Domains/Processes Made Explicit
Images Saved images can be added to the workspace . Students can use the built-in screen capture capabilities of their device to save images of their work for assessment or for additional work.	An image of the problem that a student is working on can be imported right into the tool. Images of related objects can be imported and copied (e.g., images of household objects for the mathies Money tool)	**Memory** By importing an image of the task into the mathies tool's workspace, students can more easily refer to it. They are less likely to lose track of requisite information as they move back and forth between the task and the solution. Images can be further enhanced with annotations. *Example:* A student tracks each step of the problem as they work through it with a yellow highlight and underlines important parts of the text in red.  In a bike-a-thon, cyclists will find • water stations every two-thirds of a kilometre • medical stations every three-halves of a kilometre • bike repair stations every three-fourths of a kilometre John has reached the first medical station, Alexa is at the second water station, and Sima is at the first bike repair station. Who is furthest along the course?

Resources

1. **Paying Attention to Fractions**
   http://www.edu.gov.on.ca/eng/literacynumeracy/LNSAttentionFractions.pdf

2. **Paying Attention to Spatial Reasoning**
   http://www.edu.gov.on.ca/eng/literacynumeracy/LNSPayingAttention.pdf

3. **Supporting Students with Learning Disabilities in Mathematics (York Catholic DSB Resource)**
   http://www.edugains.ca/resourcesSpecEd/IEP&Transitions/BoardDevelopedResources/IEP/SupportGuides/Supporting_Students_Learning_Disabilities_Mathematics_Ycdsb.pdf

4. **Math Teaching for Learning: Developing Proficiency with Partitioning, Iterating and Disembedding**
   http://www.edugains.ca/resources/ProfessionalLearning/Fractions_OnePageSynopses/Fractions_IteratingandPartitioning.pdf

5. **Problems Involving Addition and Subtraction**
   http://www.edugains.ca/resourcesMath/CE/LessonsSupports/WholeNumbers/AdditionandSubtractionProblems_AODA.pdf

6. **Fraction Strips Tips Sheet**
   http://mathies.ca/files/tipSheets/FractionStripsLearningToolTipsSheet.pdf

7. **Foundations to Learning and Teaching Fractions: Addition and Subtraction**
   http://www.edugains.ca/resourcesMath/CE/LessonsSupports/Fractions/FoundationstoLearningandTeachingFractions.pdf

8. **Math for Teaching: Ways We Use Fractions**
   http://www.edugains.ca/resourcesMath/CE/LessonsSupports/Fractions/SupportDocs/Fractions_WaysWeUseFractions.pdf