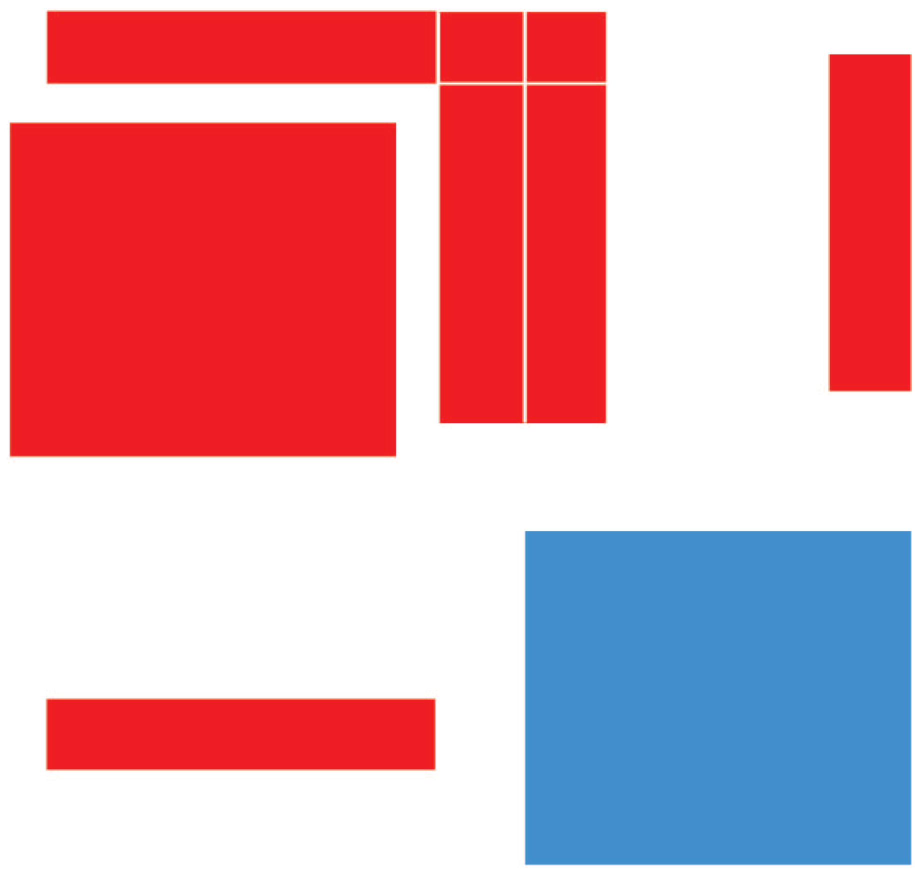
A Concrete Introduction to the Abstract Concepts of Integers and Algebra using Algebra Tiles



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Introduction

This resource is organized into two parts: Integers and Algebra. The class works through one or more of the lessons per day cooperatively, depending on the length of the classes and on student readiness. Students first use the small squares from the algebra tiles to explore integer topics. They move on to other areas of their math course and return later to build on the knowledge and familiarity with the manipualtives gained in the study of integers to develop skills in algebra.

The task is meant to start with a short teacher-guided introduction. Use manipulatives (along with an overhead or an IWB) and oral questioning at this stage. Blank out some of the entries in the charts shown with the lessons, then make a copy to use with the class on the overhead or the IWB. Ask students for the appropriate entries and fill those in during the lesson. Textbook assignments or worksheets provide students with opportunities to practise concepts. Students draw conclusions from the patterns explored and these conclusions can be polished and made into summary notes.

The importance of using the algebra tiles is to give students a visual, hands-on way of exploring patterns at the introductory stage for a new concept. This hands-on approach, using a narrative style for explaining situations, allows students to state the rules of integers and algebra from their own experience as they communicate with each other. Once students have seen the patterns and stated the rules, the tiles can be put aside. Students then extend rules to more complicated examples using only the symbolic form. It is not intended that complex examples be modelled. By the time students work with the complex examples, the concept should be understood intuitively.

The concrete → pictorial → symbolic sequence promotes understanding of algebra concepts that often elude students if only the symbolic stage is used. The **concrete** stage consists of manipulating shapes. The results will be described by whole numbers, shape, size, and colour. The **pictorial stage** is the drawing of the manipulatives. The **symbolic stage** is the written form that uses the symbols of mathematics +, – , ×, ÷, x, x^2, x^3, …, 1, 2, 3, …

Although it takes more time to include the concrete and pictorial stages, students understand the concepts at the intuitive level, which saves the time often devoted to drill, review, and reteaching later. By leading the students, through careful ordering of examples and questions, to the point where they state the patterns being followed in an operation, students 'own' the new concepts, and feel empowered to learn mathematics on their own. Their images of themselves as independent learners of mathematics becomes more positive.

When using a model, it is important to be consistent in all of the situations in which you use a model. To help develop a consistent use of the algebra tiles model, there are suggested Guiding Questions. (Break the question into smaller pieces if no answer is forthcoming from the class.) Refer students back to the model for clarification or to have them add more detail to their first responses. It is important to structure and time the questions so that students give the answers that have been suggested. This process results in the students' owning of the progress in the class. They see that the results of symbolic situations are logical and based on the situations they have modelled. Once students verbalize the appropriate pattern, write the pattern down so that they can make a note of the ideas involved.

Summary of Concepts

Integers

- The zero principle: an equal number of positive and negative tiles add to give zero. There are an infinite number of forms of zero.
- When adding, put tiles onto the table.
- When subtracting, remove tiles from the table.
- If the required tiles are not there to be subtracted, add the form of zero that will allow the subtraction.
- Once you know how to add both positive and negative integers, there is no need to use the operation of subtraction.
- Interpret multiplication questions as repeated additions, using the words "negative" and "opposite," as needed.

Algebra

Adding Polynomials

- Add like terms (tiles) using the rules of integers to get the coefficients. The shape of the tiles determines the type of term x, x^2, or unit.

Multiplying Polynomials

- Set up the dimensions of the rectangle, using one factor as the length and the other factor as the width.
- Establish the outside tiles.
- Complete the rectangle.
- Read the area of the rectangle as the result, or product.

Factoring

- Select the tiles which represent the given product, i.e., area.
- Make a rectangular array of the tiles by placing square tiles in the upper left corner and unit (1) tiles in the lower right corner.
- Read the dimensions (factors) of the completed triangle.

Solving Equations in One Variable

- Model the left and right side of the equation.
- Perform the same operation to both sides of the equation.
- Work so that the smaller number of variables is removed from both sides of the equation and work to isolate the variable.

1: Introduction to Integers

Introduce the concept of integers through the students' real-life experience with situations involving thermometer readings, sea-level, profit and loss, etc.

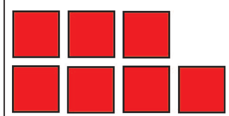
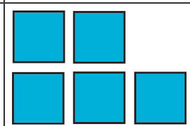
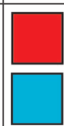
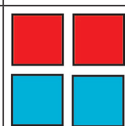
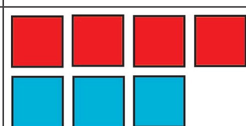
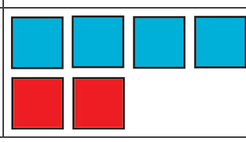
Introduce the +4 and –5 notation. Use a variety of types of questions to tie real-life situations to an integer measures.

Use purchased algebra tiles or provide students with the materials to create their own set of tiles. Using the small coloured squares from the sets of algebra tiles to teach integers provides a foundation for algera concepts. Algebra tiles for the overhead projector or IWB are also useful.

Note: Tiles come in various colours. This resource uses red and blue.

2: Working with Algebra Tiles

Work through a series of examples involving the concrete, pictorial, and verbal stages suggested in the chart below. Use only the small blue and red squares from the set of algebra tiles.

Show	Pictorial Form	Result
i) 3 red and 4 red		Red
ii) 2 blue and 3 blue		Blue
iii) 1 red and 1 blue		Is the result of combining these red or blue? (Neither)
iv) 2 red and 2 blue		Neither red nor blue
v) 4 red and 3 blue		The three pairs of red and blue give a result that is neither red nor blue. The extra red tile gives a red result.
vi) 4 blue and 2 red		blue

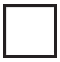
Guiding Questions

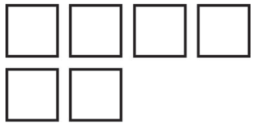
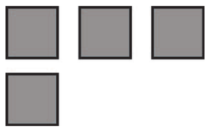
- We are getting different amounts of red or blue. How could we distinguish among the various amounts of red or blue possible in a result? (We could include numbers in our results.)
- What are the results of the above questions, using number as well as colour? (i) 7 red ii) 5 blue iii) neither iv) neither v) 1 red vi) 2 blue)

Continue with similar examples until students are able to give results by number and colour.

Small groups complete questions such as those on the Worksheet, using manipulatives.

Worksheet: Working with Algebra Tiles

Show 1 red tile as  in pictorial form and 1 blue tile as 

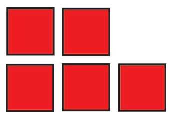
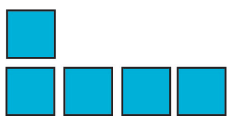
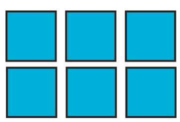
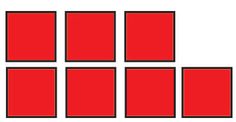
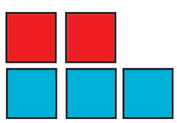
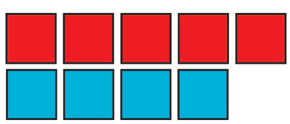
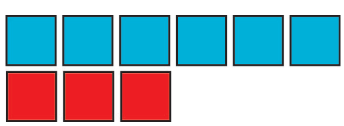
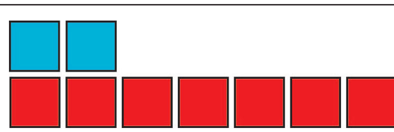
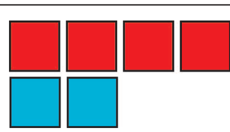
Use your tiles to model	Pictorial Form	Result by Number and Colour
4 red and 2 red		6 red
3 blue and 1 blue		4 blue
3 red and 3 blue		
5 red and 2 blue		
2 red and 3 blue		
4 blue and 3 red		
2 blue and 4 red		
4 red and 4 blue		
		2 blue
		neither red nor blue

3: Addition of Integers

Using a red tile as +1, a blue tile as – 1, and interpreting “neither red nor blue” as 0, investigate the patterns in addition of integers questions.

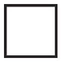
Guiding Questions

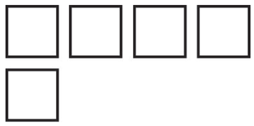
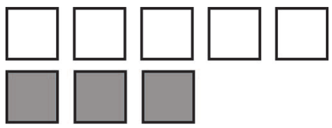
- Two types of tiles to represent the integers. How could you divide the integers into two distinct sets? What are the different types of numbers contained within the integers? (If each red tile represents the integer positive one and each blue tile represents the integer negative one, you will be able to manipulate the tiles to see the patterns involved in combining integers.)
- How should you interpret a result that is neither red nor blue; what integer is neither positive nor negative? (Zero)

Symbolic Form of the Question	Pictorial Form	Result in Number and Colour of Tiles	Result in Symbol Form
(+2) + (+3)		5 red	+5
(−1) + (−4)		5 blue	−5
(−3) + (−3)		6 blue	−6
(+4) + (+3)		7 red	+7
(+2) + (−3)		1 blue	−1
(+5) + (−4)		1 red	+1
(−6) + (+3)		3 blue	−3
(−2) + (+7)		5 red	+5
(+4) + (−2)		2 red	+2

Once students are comfortable adding integers, groups of two or three complete the Worksheet, using manipulatives.

Worksheet: Addition of Integers

Using a  tile as +1, a  tile as –1, and interpreting "neither red nor blue" as 0, complete the following chart.

Symbolic Form	Pictorial Form	Result in Number and Colour of Tiles	Result in symbols
(+4) + (+1)		5 red	+5
(−2) + (−3)			
(+5) + (−3)		2 red	+2
(−2) + (+4)			
(−3) + (+3)			
(−4) + (+1)			
(+5) − (−4)			
(+3) + (+4)			
(+2) − (−2)			
		neither red nor blue	

Answer the questions by studying the results from the table above.

1. What patterns are in the questions where you are adding two positive integers or two negative integers?

Worksheet: Addition of Integers (continued)

2. What patterns are in the questions where you are adding a positive integer and a negative integer?

3. a) What type of question leads to the answer zero?

   b) What do you have to add to 3 to generate a result of 0?

   c) What do you have to add to –7 to generate a result of 0?

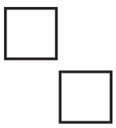
   d) Fill in the blanks. In each of b) and c) we call the number needed the "opposite" of the number given. i.e., Opposites add to give zero. The opposite of 6 is –6, of 7 is –7, of 20 is ____, and the opposite of 99 is ____. In each case you are, symbolically, putting a – sign in front of a number to form its opposite. To be consistent, we should form the opposite of –5 by placing a – sign in front of it. We get – (−5). But we know that the opposite of –5 is _____ since 5 adds to –5 to give 0. Therefore – (−5) = 5. We read "the opposite of negative 5 is five."

   e) What are the two different ways in which we read a – sign in part d?

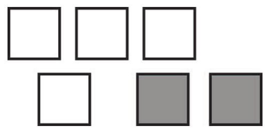
Worksheet: Addition of Integers (continued)

4. What integer result do you get from evaluating the following combinations of tiles?

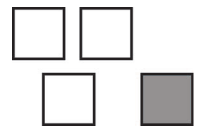
a)



b)



c)



5. Use your results from Question 4 to show three different pictorial representations of the integer –3.

6. a) What is the minimum number of tiles that could be used to model the integer +1?

   b) Show a model of +1 that uses five tiles.

7. a) If you had the model of any integer and added one red tile and one blue tile to your model, would the resulting integer be the same or different from the original integer? Explain.

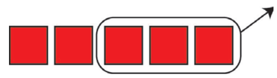
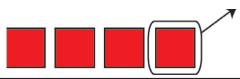
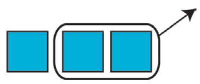
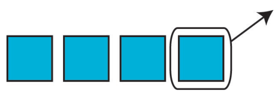
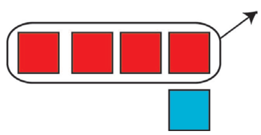
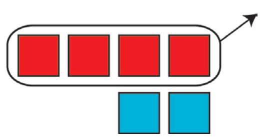
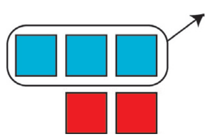
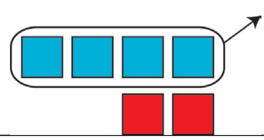
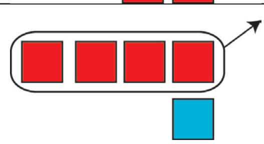
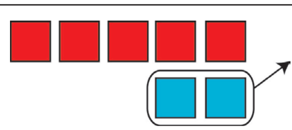
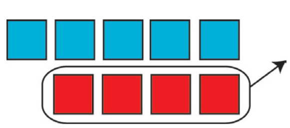
   b) If you added two red and two blue tiles to the model of any integer, how would the resulting integer compare to the original integer? Explain.

4: Subtraction of Integers

Investigate patterns in subtraction of integers questions.

Guiding Questions

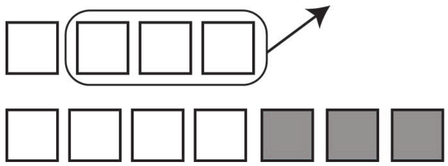
- What is the meaning of subtraction? (Take away, reduce, remove, the opposite of addition.)
- How could we use tiles to model subtraction? (Start with the model for the first integer and take the model of the second integer away.)

Symbolic Form of the Question	Pictorial Form	Result in Number and Colour of Tiles	Symbolic Result
(+5) – (+3)		2 red	+2
(+4) – (+1)		3 red	+3
(−3) – (−2)		1 blue	−1
(−4) – (−1)		3 blue	−3
(+3) – (+4)		1 blue	−1
(+2) – (+4)		2 blue	−2
(−1) – (−3)		2 red	+2
(−2) – (−4)		2 red	+2
(+5) – (−1)		6 red	+6
(+3) – (−2)		5 red	+5
(−1) – (+4)		5 blue	−5

Once students are giving comfortable subracting integers, small groups complete the Worksheet, using manipulatives.

Worksheet: Subtraction of Integers (with comparison to adding integers)

Using a red tile as +1, a blue tile as –1, and a combination of an equal number of red and blue tiles as 0, complete the following chart. To subtract, you take away tiles. To add, you put tiles together.

Symbolic Form	Pictorial Form	Result in Number and Colour of Tiles	Result in symbols
i) (+4) − (+3) (+4) + (−3)		1 red 1 red	+1 +1
ii) (−3) − (−1) (−3) + (+1)			
(iii) (+2) − (+5) (+2) + (−5)			
(iv) (−5) − (+1) (−5) + (−1)			
v) (+3) − (−2) (+3) + (+2)			

Worksheet: Subtraction of Integers (with comparison to adding integers) (continued)

1. Notice that each subtraction question has an addition question paired with it.
   a) What do you notice about the first integer in each pair of questions?

   b) What do you notice about the results in each pair of questions?

   c) In pair i), what is the effect of subtracting positive three as compared to that of adding negative three?

   In pair iii) we see that subtracting +5 is the same as __________

   d) What overall pattern can you see from your results in c)?

2. a) If you have the model of any integer and from it take away an equal number of red and blue tiles, how will your result compare to the original integer?

   b) What does this say about subtracting zero from any integer?

3. In our work to date, we have interpreted a – sign in three different ways, depending on the context:
   In (+4) + (−3) we say "positive four add negative three"
   In "What is the symbolic way of writing the opposite of 5?", we show –5
   In 5 − 2 we say "5 subtract 2."
   a) What would be three different ways of interpreting –4 in a question, depending on the context?

   b) What would be three different ways of interpreting – (−2) in a question, depending on the context?

   c) What other symbols would have the same meaning as – 2(−2) in any question?

5: Mathematical Shorthand

- You have written positive integers with + signs in front and negative integers with – signs in front. You have also used the + sign to mean the operation of addition and the **–** sign to mean the operation of subtraction. Now that you can add and subtract positive and negative integers, it is time to look at mathematical shorthand.
- On the Worksheet: Subtraction of Integers you noticed that we get the same result in (+4) **–** (+5) as we do in (+4) + (**–**5). Mathematicians agree that the symbols +4 **–** 5 can be read as either "positive four subtract positive five" or as "positive four add negative five."

  Also the questions (+5) **–** (+3) and (+4) + (+1) in integer form are the same as the questions 5 **–** 3 and 4 + 1. When an integer is positive, the + sign can be understood rather than written. Therefore, the questions (+4) **–** (+5) or (+4) + (**–**5) are normally written 4 **–** 5. With no sign in front of it, we assume that the 4 is positive. Remember that the "**–**5" part can be read as "subtract positive 5" or as "add negative 5."
- It will depend on the context of a question which way you choose to read a – sign. e.g., – 2−3 could be expressed as (i) negative two add negative three, or (ii) negative two subtract positive three, or (iii) negative two subtract three, or (iv) minus two minus three.

Each of these ways of reading the question should bring a mental image of the tiles and how to combine them. For most people, the addition model is easier to picture in their minds. For that reason, we will refer to the addition model rather than the subtraction model in future questions. Subtraction is not needed as an operation now that we know how to add a positive or a negative integer to any other.

Show longhand forms and ask for the shorthand equivalent form and the result. Students suggest longhand forms and results for the shorthand equivalent.

Longhand Form	Shorthand Equivalent	Result
(+4) + (−3)	*4 – 3*	*1*
(+5) – (+2)	*5 – 2*	*3*
(−3) + (+7)	*−3 + 7*	*4*
(−2) + (−8)	*−2−8*	*−10*
(+10) – (+15)	*10 – 15*	*−5*
(+12) – (+7)	*12 – 7*	*5*
0 + (−5)	*0−5*	*−5*
(+9) + (+6)	*9 + 6*	*15*
(+8) – 0	*8 – 0*	*8*
(+8) + (−5) + (−2)	*8 – 5 – 2*	*1*
(−10) – (+8) + (+7)	−10 – 8 + 7	−11
	10 – 12	
	−6 + 20	
	−5−4	
	9−7	

Once students are familiar with manthematical shorthand, they complete assigned textbook exercises.

6: Multiplication of Integers

Recall

- A quick way of doing repeated additions is to multiply.

  e.g., $5 + 5 + 5 + 5 + 5 + 5 + 5 = 7 \times 5$. Add seven groups of five.

  $3 + 3 + 3 + 3 = 4 \times 3$ Add four groups of three.

  $2 + 2 + 2 = 3 \times 2$ Add three groups of two.

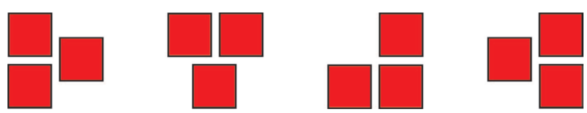
- The order in which you multiply does not matter.

  e.g., $4 \times 3 = 3 \times 4$ $5 \times 7 = 7 \times 5$ $9 \times 2 = 2 \times 9$

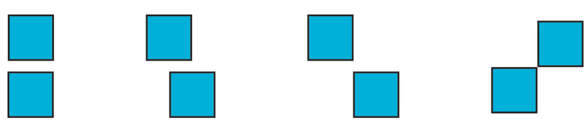
- There are three ways of reading a – sign depending on the context.
  They are: “negative,” “opposite of,” “subtract.”

Use all of these ideas to investigate the patterns in the multiplication of integers.

1. 4(3) means the same as adding 4 groups of 3. In the concrete model, you start with zero and put four groups of three red tiles on the table.



2. Similarly, 5(−2) means add five groups of negative two. The model would have five groups of two blue tiles.



   (−2)(5) may seem harder to visualize than the example above. However, once we recall that the order of multiplication does not matter, we see that (−2)(5) = (5)(−2) = – 10.

   Another way of looking at (−2)(5) is to use the “opposite” interpretation of the – sign. We say that (−2)(5) is “the opposite of two times five.” We know that $2 \times 5 = 10$ and that the opposite of 10 is – 10.

3. A third way of looking at (−2)(5) could include the interpretation “subtract” for the – sign. That is, (−2)(5) is “zero subtract two groups of five.” Recall from the subtraction model that, if you are to remove groups of five from the model, you must have those groups of five present to start with. But, you start with zero. How is this possible? We use the zero principle that says: if we have an equal number of red tiles and blue tiles to start, then we are starting with zero. The questions would then look like:

6: Multiplication of Integers (continued)

4. A fourth way of considering (−2)(5) would be to look at the pattern development through comparing one row, both question and answer, to the next row in this chart:

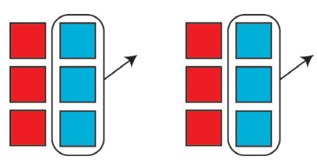
<table>
<tr><td rowspan="6">You take one more factor of five in each row than you had in the row above.</td><td>(3)(5) = 15</td><td rowspan="6">You subtract five from the result in one row to get the result for the next row. Therefore, (−1)(5) = – 5 and (−2)(5) = – 10</td></tr>
<tr><td>(2)(5) = 10</td></tr>
<tr><td>(1)(5) = 5</td></tr>
<tr><td>(0)(5) = 0</td></tr>
<tr><td>(−1)(5) = ?</td></tr>
<tr><td>(−2)(5) = ?</td></tr>
</table>

Note: Point out that no matter which of the interpretations we use, we are consistently getting the same result.

You can now interpret (−2)(−3) using any of the patterns above:

i) (−2)(−3) could be read "the opposite of two times negative three" to give the opposite of negative six which is positive six.

ii) (−2)(−3) could read "subtract two groups of negative three." This interpretation makes sense only if you think of "zero subtract two groups of negative three." To show this with a model, we would start with six red tiles and six blue tiles (which together have a value of zero), then we would take away two groups of three blue tiles. Can you envision the six red tiles being left as your result? If not, model the scenario with your tiles. We again get the result positive six.



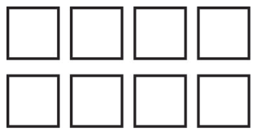
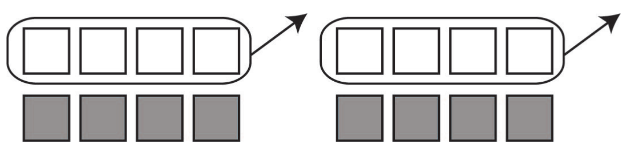
iii) You can get to the question (−2)(−3) by working through these patterns in the questions and answers:

(3) (−3)	=	−9
(2) (−3)	=	−6
(1) (−3)	=	−3
(−1) (−3)	=	?
(−2) (−3)	=	?

Once students understand the ways of investigating the multiplication of integers, groups discover the overall patterns using the Worksheet and manipulatives.

Worksheet: Multiplication of Integers

1. Complete the chart:

Symbolic Form of Multiplication	Pictorial Form	Symbolic Result
a) (+4)(+2)		8
b) (+2)(−3)		
c) (−1)(+2)		
d) (−2)(−4)	 Or the opposite of: 	8
e) (+5)(−1)		
f) (−3)(−2)		

Worksheet: Multiplication of Integers (continued)

2. Complete the chart:

Multiplication Question	Result	Signs in Question and Result
a) (+2)(+6)	+12	(+) (+) = (+)
b) (+5)(+7)		
c) (+3)(−4)		
d) (+6)(−7)		
e) (−6)(−2)		
f) (−3)(−9)		
g) (−6)(−1)		
h) (−6)(−8)		

3. Use your results in questions 1 and 2 to state a pattern between the signs in a multiplication question and the sign of the answer.

4. Complete the chart:

Signs in a Multiplication Question	Sign of the Result
a) (+)(+)(+)	(+)
b) (+)(+)(−)	(−)
c) (+)(−)(−)	
d) (−)(+)(−)	
e) (−)(−)(+)	
f) (−)(−)(−)	
g) (+)(+)(+)(+)	
h) (+)(−)(−)(−)(−)	
i) (−)(−)(−)(−)(−)	
j) 6 negatives	
k) 7 negatives	
l) 8 negatives	
m) an odd number of negatives	
n) an even number of negatives	
o) any number of positives	

5. How could you generalize the pattern for deciding the sign of the result of multiplication of integers?

7: Division of Integers

Recall

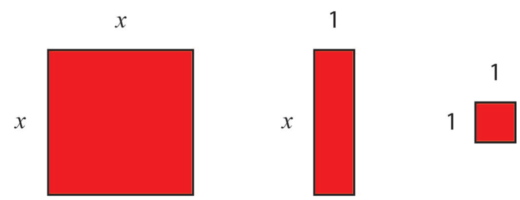
Recall that division is the inverse operation of multiplication. When you are given a question like $18 \div (-3)$, you could ask what integer multiplies by (-3) to give 18.

Assign textbook exercises.

8: Modelling Polynomials with Algebra Tiles

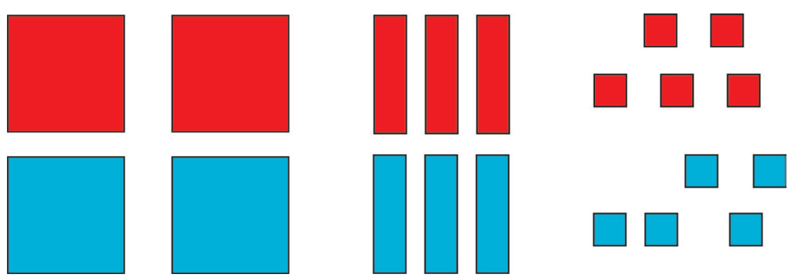
Use the full set of algebra tiles for the work on polynomials. The sets of tiles can be used to model polynomials of the form $ax^2 + bx + c$ or of the form $ax^2 + bxy + cy^2$, where a, b, and c are constants.
Note: Model only polynomials of the form $ax^2 + bx + c$ to establish the appropriate concepts. Students will make the extension to more complex examples at the abstract level, once the concepts are clearly understood.

Show that each large square has a length of x units and a width of x units, each rectangle has a length of x units and a width of one unit, and each small square has a length of one unit and a width of one unit.
Note: no integral number of unit lengths generates a measure equal to the x measure.



Since the three shapes have different sizes, consider what symbolic forms would be consistent with the measurements of the shapes. When we compare two-dimensional shapes, we refer to the area of the shape when we say that one shape is larger than another. The area of each large square is $(x)(x)$ or x^2; the area of each rectangle is $(x)(1)$ or x; the area of each small square is (1)(1) or 1. It would be reasonable, to let each large square model a term of size x^2, each rectangle a term of size x, and each small square a term of size 1.

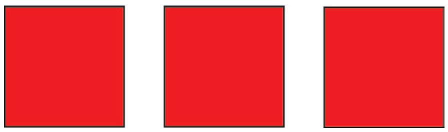
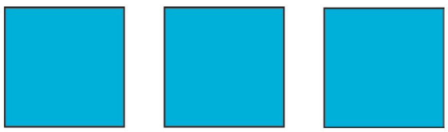
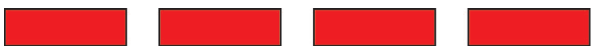
Recall the zero principle and extend it to all three shapes. Each of the following combinations of tiles would give a result that is neither red nor blue. Therefore, each of these combinations is a model for zero.



As with the integer tiles, we have both red and blue tiles in each of the three sizes. Recall that if we combine more red than blue tiles of the same size together, our result is red. If we combine more blue than red tiles, our result is blue.

8: Modelling Polynomials (continued)

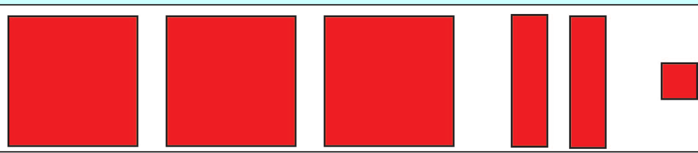
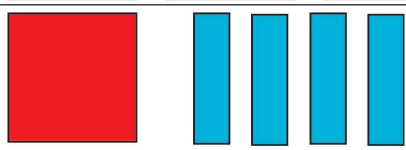
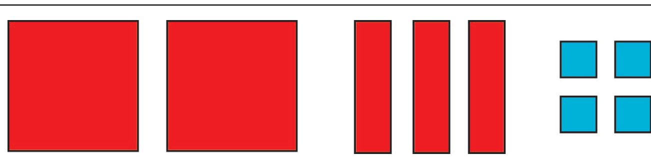
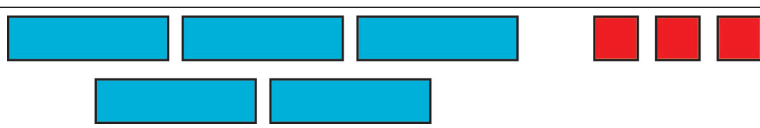
Work through a series of examples involving the concrete, pictorial, and verbal stages suggested:

Pictorial Form	Result by Number and Colour and Size
i) 	2 red small squares
ii) 	3 red large squares
iii) 	3 blue large squares
iv) 	4 red rectangles

Guiding Question

- How could you record the results symbolically? i) 2   ii) $3x^2$ iii) $-3x^2$ iv) $4x$)

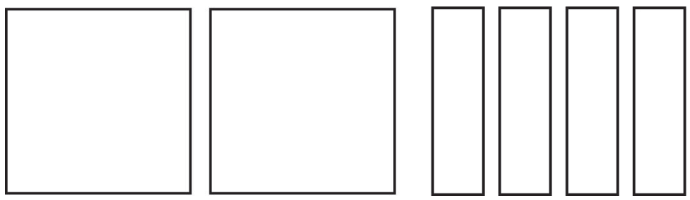
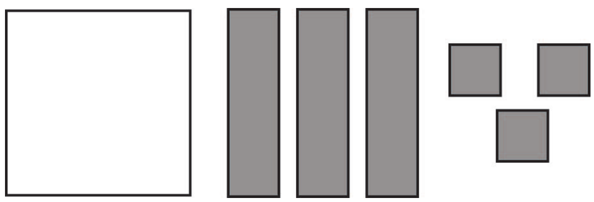
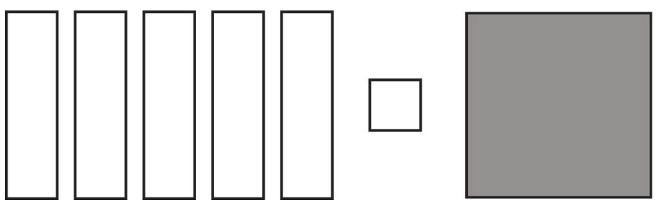
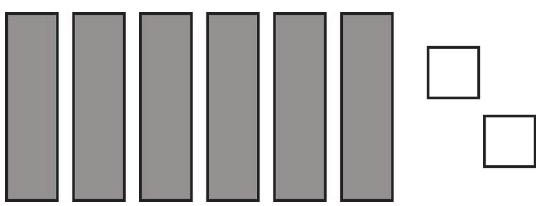
Once students are ready, advance to examples such as:

Pictorial Form	Symbolic Form
	$3x^2 + 2x + 1$
	$x^2 - 4x$
	$2x^2 + 3x - 4$
	$-5x + 3$

Once students are comfortable modelling polynomials, small groups complete the Worksheet, using manipulatives.

Worksheet: Modelling Polynomials

1. Complete the chart:

Pictorial Form	Symbolic Form
a) 	$2x^2 + 4x$
b) 	
c) 	
d) 	
e)	$5 - 6x$
f)	$x^2 + 3x - 1$
g)	$x - 3x^2$
h)	$-2x^2 + 8$

Worksheet: Modelling Polynomials (continued)

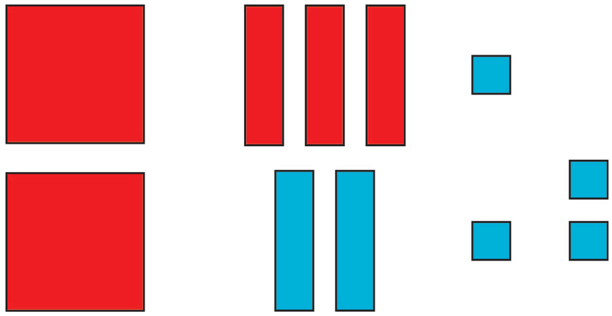
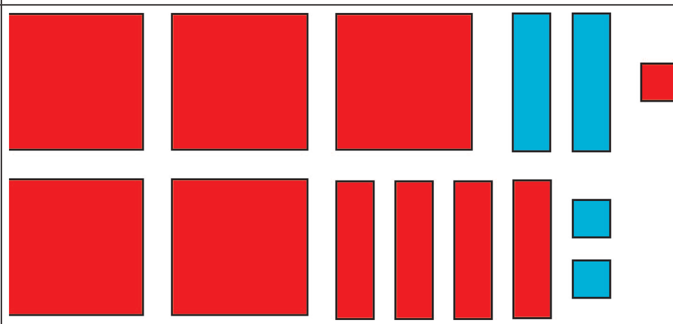
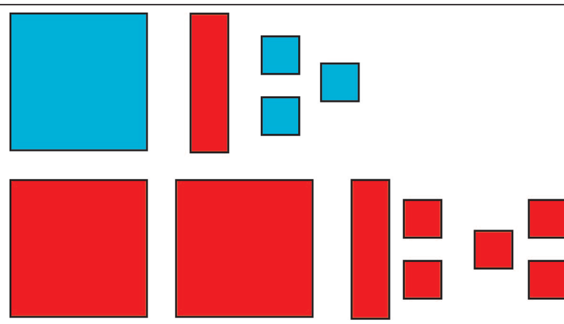
2. a) What is the minimum number of tiles that could be used to model $x^2 + 3x - 1$? Show your model.

   b) Add two tiles to your model in a) so that you are still modelling $x^2 + 3x - 1$.

   c) Show six different models of $x^2 + 3x - 1$ that use nine tiles each.

9: Addition of Polynomials

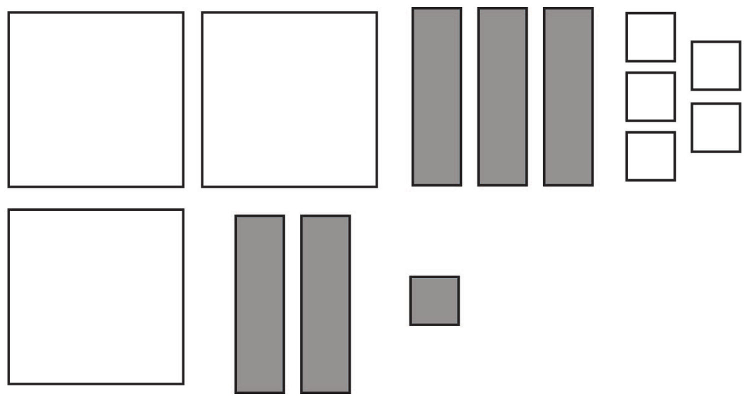
Work throught a series of examples.

Symbolic Form of Two Polynomials	Pictorial Form	Symbolic Result
$x^2 + 3x - 1$ $x^2 - 2x - 3$		$2x^2 + x - 4$
$3x^2 - 2x + 1$ $2x^2 + 4x - 2$		$5x^2 + 2x - 1$
$-x^2 + x - 3$ $2x^2 + x + 5$		$x^2 + 2x + 2$

Once students are comfortable adding polynomials, they complete the Worksheet, using manipulatives. After discussing their results, they completes textbook exercises.

Worksheet: Addition of Polynomials

Use your tiles to model the additions and complete the chart:

Symbolic Form	Pictorial Form	Symbolic Result
$2x^2 - 3x + 5$ $x^2 - 2x - 1$		$3x^2 - 5x + 4$
$x^2 + 4x - 6$ $x^2 - 2x - 1$		
$5x - 3$ $-2x + 1$ $x - 2$		
$7 - 3x + x^2$ $-4 - x - x^2$		

Use the results to answer:

1. How do we add polynomials together, if we use only the symbolic form?

Worksheet: Addition of Polynomials (continued)

2. The tiles that have the same size and share area like; in symbolic form we call them like terms. Reword your observations in Question 1 using the phrase "like terms."

3. A term is a mathematical symbol that contains a numerical and/or a literal part. The term $7x^2$ has numerical coefficient 7 and literal coefficient x^2. Term $-6abc$ has numerical coefficient –6 and literal coefficient abc.

   Use these examples to help choose the appropriate entries for the chart:

Term	Numerical Coefficient	Literal Coefficient
$5x^3$		
$-3y^4$		
$82x^2y^5$		
	−99	p^2

Each term has a degree that depends on the number of literal factors it has. Term $5x^3$ has degree 3; term $-3y^4$ has degree 4; term $82x^2y^5$ has degree 7.

Use these examples to help you complete the chart:

Term	Numerical Coefficient	Literal Coefficient	Degree of the Term
$4a^5$	4	a	5
$-5c^4d^3$			
$24x^8y$			
$8a^2b^4$			
$-6c^5de^3$			

Worksheet: Addition of Polynomials (continued)

4. In the symbolic form, addition questions are often written horizontally, rather than vertically.

   e.g., $(5x^2 + 3x - 7) + (3x^2 - 4x + 2)$ means that we are to add the two bracketed polynomials. We add $3x^2$ to $5x^2$, $-4x$ to $3x$, and 2 to -7. We could communicate this question with fewer symbols if we wrote $5x^2 + 3x - 7 + 3x^2 - 4x + 2$. In a question written in string form like this, it is easiest to think of collecting like terms if we visually isolate two like terms, remembering to take any sign in front of the term as a positive or negative sign, and add.

   When we simplify $5x^2 + 3x - 7 + 3x^2 - 4x + 2$, we get $8x^2 - x - 5$. Model this question with tiles, to verify the result.

   Simplify the following questions.

   a) $2x^2 + 7x + 8x^2 - x - 6$

   b) $3x^2 - 6x^2 - 4x^2 + 7x^2$

   c) $10ab - 5c + 2ab + 7c$

   d) $9y^4 + 7y3 - y + 8y - 5y^3 + y^4$

   e) $7pqr + 2r - 3pqr - r$

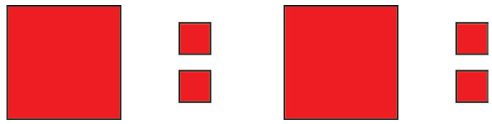
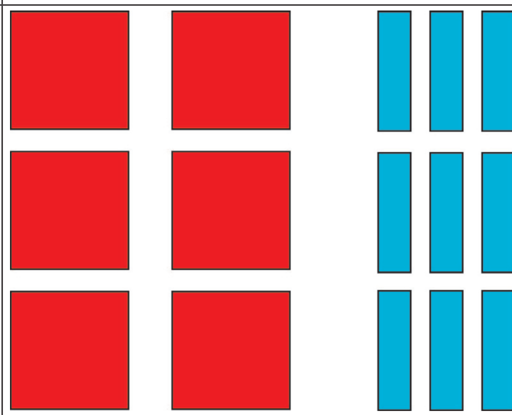
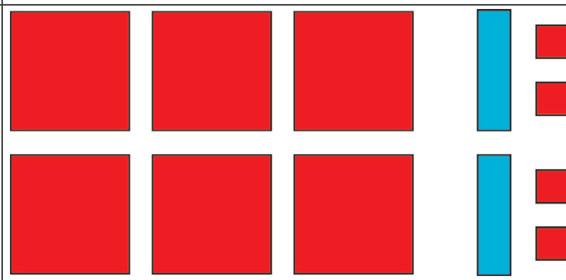
   f) $8a^2 + 4a - a^2 - 9a$

   In your own words, write the pattern you are using to generate answers in the symbolic form.

   Notice that in the questions you have just simplified, you added both positive and negative terms. Another way of saying what you have done, is that you have added and subtracted terms. So all subtraction questions can be thought of as addition questions.

10: Multiplying a Polynomial by a Monomial (Expanding)

Students model examples.

Model Polynomial	Then	Pictorial Form	Symbolic Results
$x^2 + 2$	$2(x^2 + 2)$		$2x^2 + 4$
$2x^2 – 3x$	$3(2x^2 – 3x)$		$6x^2 – 9x$
$3x^2 – x + 2$	$2(3x^2 – x + 2)$		$6x^2 – 2x + 4$

Guiding Question

- How could we get the results at the symbolic level without using the tiles? (Multiply the monomial times each of the terms of the polynomial.)

Point out that this process, which removes the brackets, is called expanding. Ask the students to expand questions such as: $-2(x + 1)$

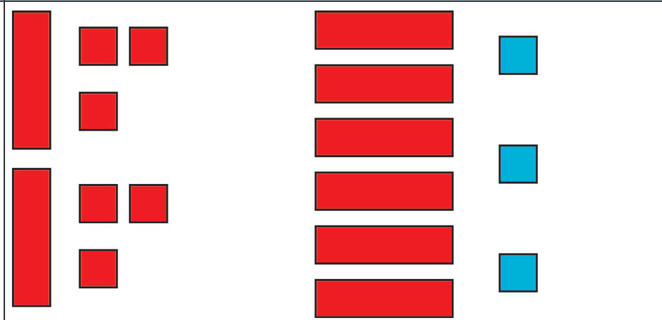
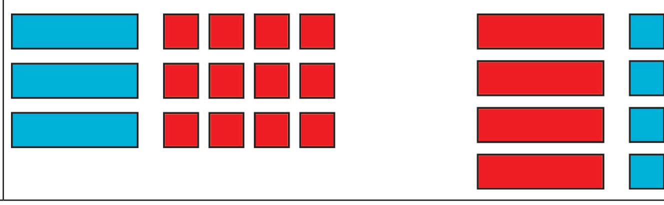
If students ask for the model of this question, remind them of the interpretation $0 – 2(x + 1)$ and show

Start with an equal number of red and blue tiles as zero. We take away 2 groups of $x + 1$. You are left with $–2x – 2$. Encourage students to use just the symbolic level in further questions of this type such as:

$-6(2x + 4)$ $\quad\quad$ $-5(-3x – 2)$

10: Multiplying a Polynomial by a Monomial (Expanding) (continued)

Once students are giving consistently correct results continue with examples, such as:

Model	Pictorial Form	Symbolic Form of the Simplification
$2(x + 3) + 3(2x - 1)$		$8x + 3$
$3(-x + 4) + 4(x - 1)$		$x + 8$

Guiding Question

- How could we do these questions symbolically without the tiles? (Expand and collect like terms.)

Simplify

1. $2(x - 3) - 3(4 - x)$ Remind students that you can read this question as "two times the first binomial add negative three times the second."

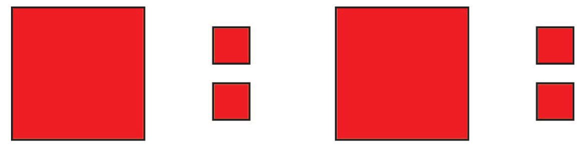
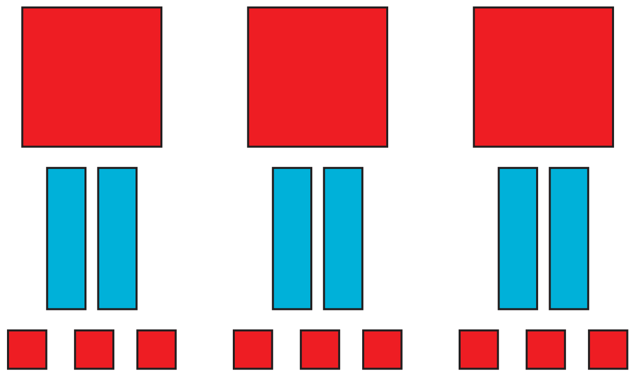
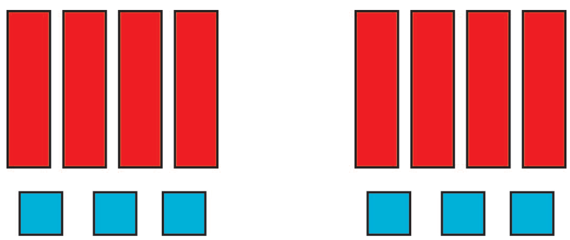
2. $5(a - 5) + (a + 2)$ Remind students that a coefficient of 1 is understood.

3. $(3b^2 - c) - (b^2 + 4c)$

Once students are able to multiply a polynomial by a monomial, they complete assigned textbook exercises.

11: Dividing a Polynomial by a Monomial

Use the tiles to show this concept visually. Students know intuitively that to divide by two means to split whatever you have into two equal groups, and to divide by three means to split whatever you have into three equal groups.

Model for Polynomial	How to Model	Pictorial Form	Symbolic Results
$2x^2 + 4$	$\frac{2x^2+4}{2}$		$x^2 + 2$
$3x^2 - 6x + 9$	$\frac{3x^2-6x+9}{3}$		$x^2 - 2x + 3$
$8x - 6$	$\frac{8x-6}{2}$		$4x - 3$

Guiding Question

- How could we do these questions symbolically without the tiles? (Each term in the numerator is divided by the monomial in the denominator.)

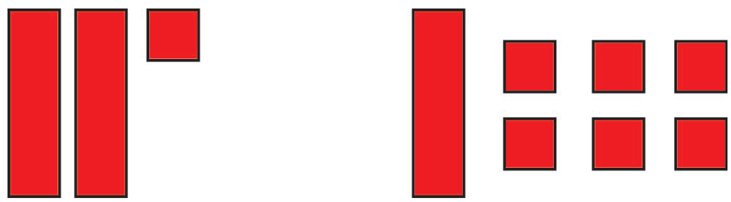
Once students are able to divide a polynomial by a monomial, assign textbook exercises.

12: Solving Linear Equations in One Unknown

Recall

- An equation is like a balance and whatever is done to one side of the equation (i.e., added to, subtracted from, multiplied by, or divided by), must be done to the other side.
- The meaning of solving an equation is: Find the value of the variable which will “satisfy” the equation, or When the value of the variable is substituted into all variable positions, both sides of the equation will be equal.
- The zero principle.

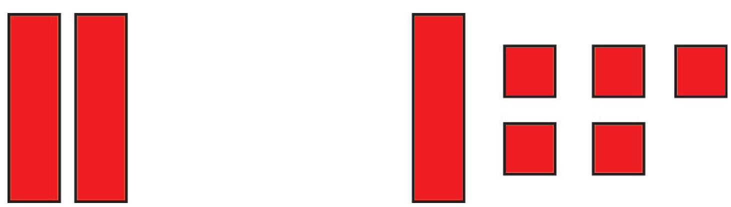
Example 1: Solve the equation $2x + 1 = x + 6$.
Show the left side of the equation with tiles. Show the right side of the equation with tiles. Create the equation using the tiles.



Guiding Question

- Assume that we wish the variable to be on the left side. What term is also on that side and how can it be removed? (The 1 can be removed by subtraction, remembering that the same must be done to the right side.)

Subtract one from both sides. The remaining equation is $2x = x + 5$.



Guiding Question

- To isolate x on one side, what must now be done to both sides? (Subtract x from both sides.)

Subtract x from both sides. The remaining equation is $x + 5$.

If $x = 5$ is substituted into both sides of the equation, both sides equal 10.
Thus, the solution of the equation is $x = 5$.

12: Solving Linear Equations in One Unknown (continued)

Example 2: Solve the equation $3x - 4 = x + 2$.
Form the equation using the tiles.

Guiding Question

- What needs to be done to both sides of the equation? (Add 4 to both sides)

Show the result of adding 4 to both sides. $3x - 4 + 4 = x + 2 + 4$ or $3x = x + 6$.

Guiding Question

- What needs to be done next? (Subtract x from both sides.)

Show the result by subtracting x from both sides. $3x - x = x + 6 - x$ or $2x = 6$.

Guiding Question

- How would you find the value of x now? (Divide both sides by 2 and obtain $x = 3$.)

The solution to equation $3x - 4 = x + 2$ is $x = 3$.

12: Solving Linear Equations in One Unknown (continued)

Example 3: Solve the equation $2x + 5 = 5x – 4$.

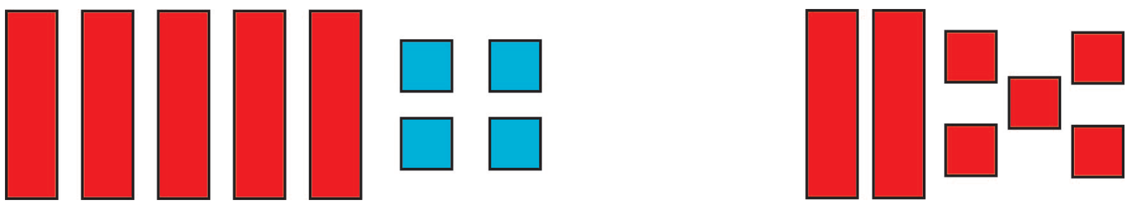
Guiding Questions

- What has to be done to both sides of the equation to isolate x? (Subtract 5 from both sides and subtract $5x$ from both sides, or add 4 to both sides and subtract $2x$ from both sides.)

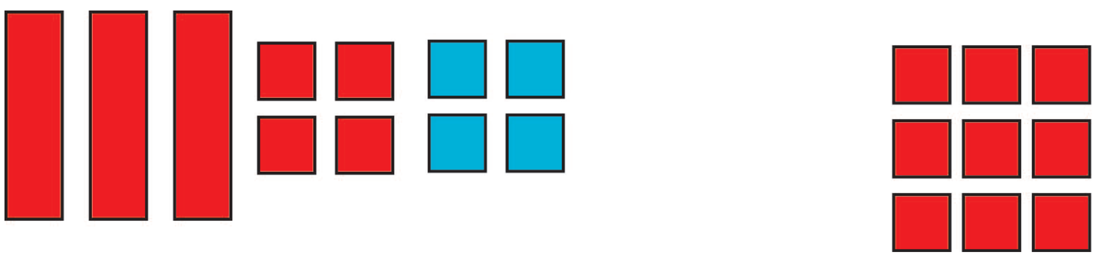
  Note: Since division by a negative number is difficult to physically illustrate, the teacher could suggest to the students that they reverse the equation to that the larger number of variables is on the left side. With the tiles, this is very easy to show. Another option is to suggest that students work from the side where the larger number of the variables occur. Once the concept of solving equations without using the tiles is developed, the above suggestions are not needed.

- Is the equation $2x + 5 = 5x – 4$ the same as the equation $5x – 4 = 2x + 5$? (Yes, because the equation is like a balance.)

Show the equation $2x + 5 = 5x – 4$ above as $5x – 4 = 2x + 5$ using the tiles.



Add 4 to both sides and subtract $2x$ from both sides. $5x – 4 + 4 – 2x = 2x + 5 + 4 – 2x$



The result shows $3x = 9$.

Guiding Question

- How do we find the value of x? (Divide both sides by 3, i.e., sort into 3 equal parts.)

The result is $x = 3$.

12: Solving Linear Equations in One Unknown (continued)

Students should be ready to omit the tiles and work at the symbolic level.

Symbolic Form	What must be done to both sides?	Result in symbols
$3x + 2 + 2x + 5$	Subtract 2 Simplify. Subtract $2x$ Simplify.	$3x + 2 - 2 = 2x + 5 - 2$ $3x = 2x + 3$ $3x - 2 = 2x + 3 - 2x$ $x = 3$
$4x - 1 = 2x - 3$	Add 1 Simplify. Subtract $2x$. Simplify. Divide by 2	$4x - 1 + 1 = 2x - 3 + 1$ $4x = 2x - 2$ $4x - 2x = 2x - 2 - 2x$ $2x = - 2$ $x = 1$

Continue with examples until students are ready to complete assigned textbook exercises without manipulatives.

13: Multiplying a Binomial by a Monomial

(**Note:** This concept is revisited to introduce Multiplication of Binomials #14)

> **Recall**
> - Multiplication of Integers
> - Meanings of Monomials, Binomials, Trinomials, Polynomials

Example 1: Find the value of $2(x + 3)$.

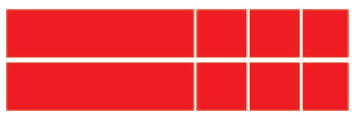
Guiding Question

- What is the meaning of $2(x + 3)$? (The expression $(x + 3)$ is to be doubled or multiplied by 2.)

Show $(x + 3)$ with your tiles.

Show $2(x + 3)$

Move the tiles together into a rectangle



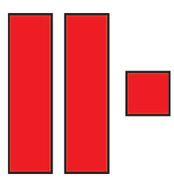
Notice that the rectangle has a width of 2 and length of $(x + 3)$, and that the area of the rectangle is $2x + 6$. Therefore $2(x + 3) = 2x + 6$ could be a model for the width times the length of a rectangle equalling the area.

Example 2: Find the value of $3(2x + 1)$.

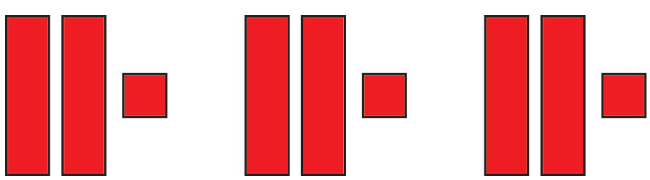
Guiding Question

- What is the meaning of $3(2x + 1)$? (The expression $(2x + 1)$ is to be tripled or multiplied by 3.)

Show $(2x + 1)$



Show $3\ (2x + 1)$



Move the tiles together into a rectangle:

The rectangle formed has the dimensions 3 and $(2x + 1)$ and has an area of $6x + 3$.

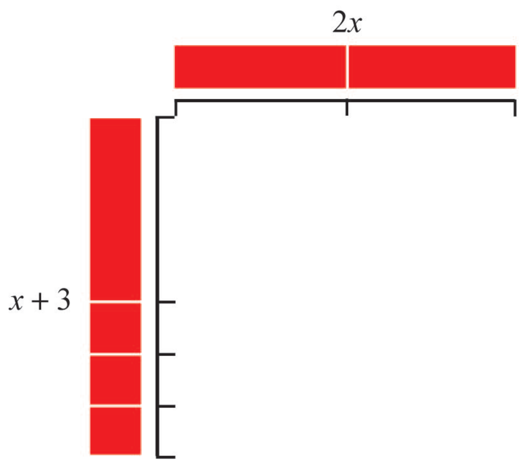
13: Multiplying a Binomial by a Monomial (continued)

Example 3: Find the value of $2x(x + 3)$.

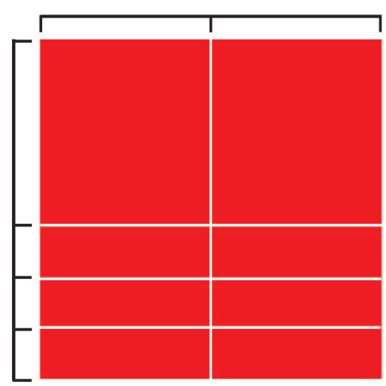
Guiding Question

- What would be the dimensions of the rectangle formed using tiles? (One dimension would be $(2x)$ and the other would be $(x + 3)$.)

Show the dimensions.



Complete the rectangle.



Therefore $2x(x + 3) = 2x^2 + 6x$.

Guiding Question

- What seems to the pattern of multiplying the monomial times the binomial? (The term in front (the monomial) is multiplied by each term in the binomial.)

Use this idea to expand a variety of questions. Once the students are able to use this model to multiply a binomial by a monomial, groups of two or three complete textbook exercises. Encourage students to refer back to the model, if necessary.

14: Multiplication of Binomials

Recall

- rules of operating with integers.
- multiplying a binomial by a monomial.

Guiding Question

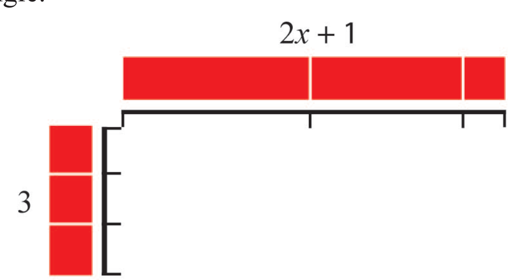
- How did we use the algebra tiles to illustrate the result of multiplying a binomial by a monomial? (We created a rectangle where the monomial was one dimension and the binomial was the other dimension. The answer was the area of the created rectangle.)

Note: This concept is further developed when a binomial is multiplied by another binomial. Once again we create a rectangle using the two factors (binomials) as the length and width of the rectangle. The expanded form is the area of the rectangle. We determine this area by forming the concrete rectangle out of the tiles. The pieces are chosen so that the resulting rectangle will have the length and width suggested by the factors. Since the area of each tile is known, the area of the rectangle will be the sum of the areas of the individual tiles used.

Recall the result of multiplying a binomial by a monomial.

Example: Fine the value of $3(2x + 1)$.

Set up the dimensions of the rectangle.



Complete the rectangle using the tiles.

When completed, this rectangle has width 3 and length $2x + 1$. Note that its area is $6x + 3$.

(length) (width) = area

Thus, $3(2x + 1) = 6x + 3$ or $(2x + 1)(3) = 6x + 3$.

Procedure

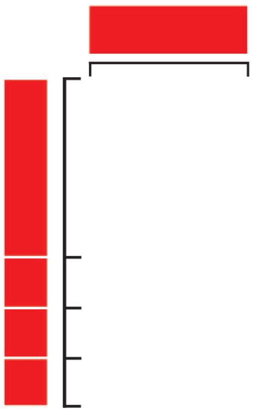
Set up the dimensions of the rectangle by placing the terms of one factor along the top of your workspace and the other factor along the side of the workspace.

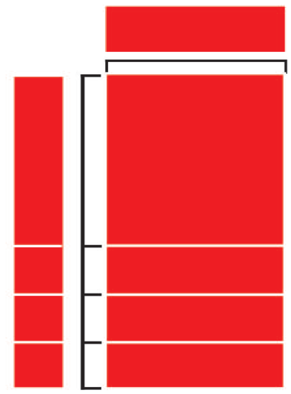
Complete the rectangle, if necessary.

State the results by assessing the area sum.

14: Multiplication of Binomials (continued)

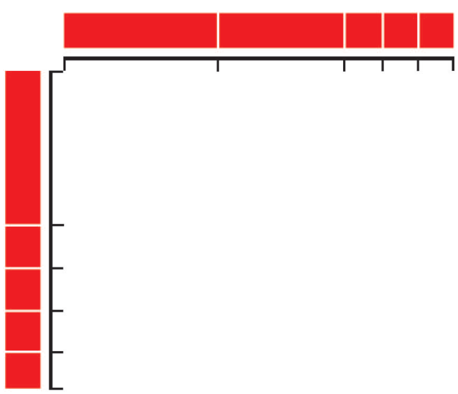
Example 1: Find the value of $(x)(x + 3)$.





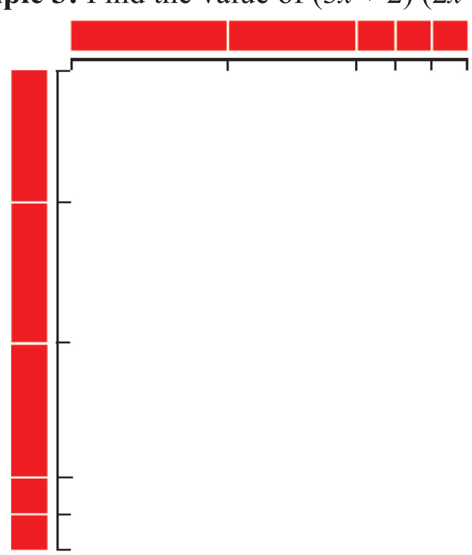
Therefore $(x)(x + 3) = x^2 + 3x$

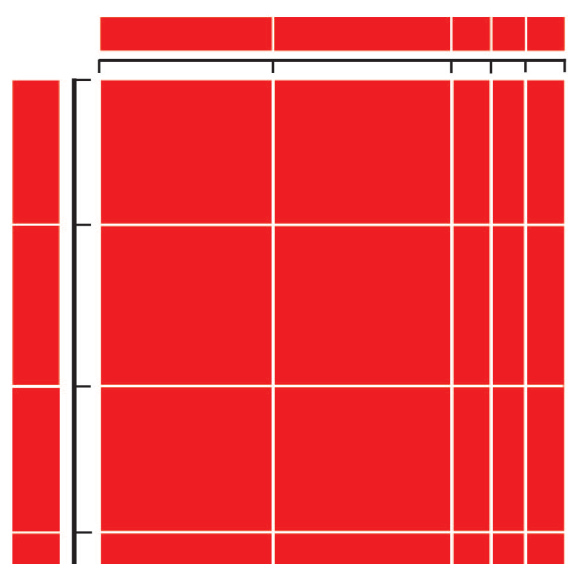
Example 2: Find the value of $(2x + 3)\ (x + 4)$.



Therefore $(2x + 3)\ (x + 4) = 2x^2 + 11x + 12$.

Example 3: Find the value of $(3x + 2)\ (2x + 3)$

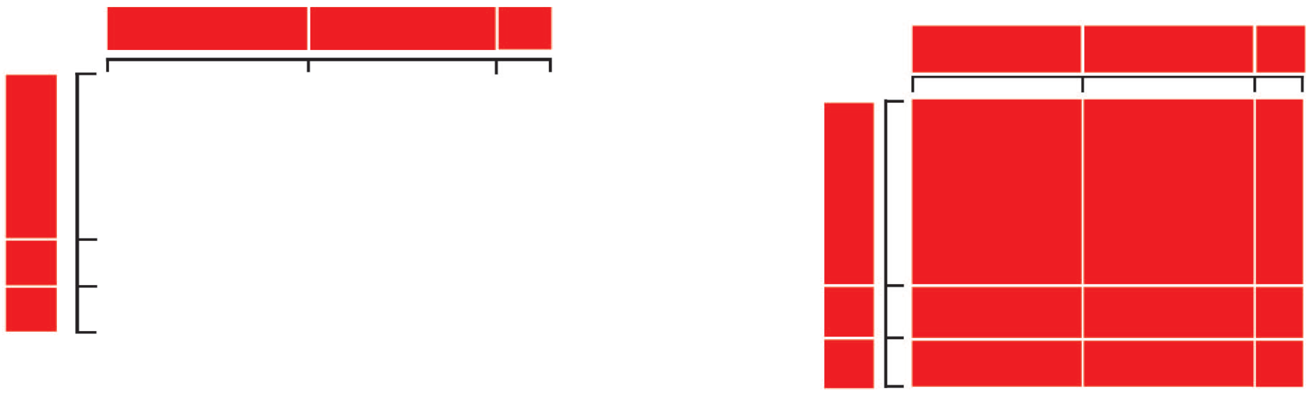




Therefore $(2x + 3)\ (3x + 2) = 6x^2 + 13x + 6$

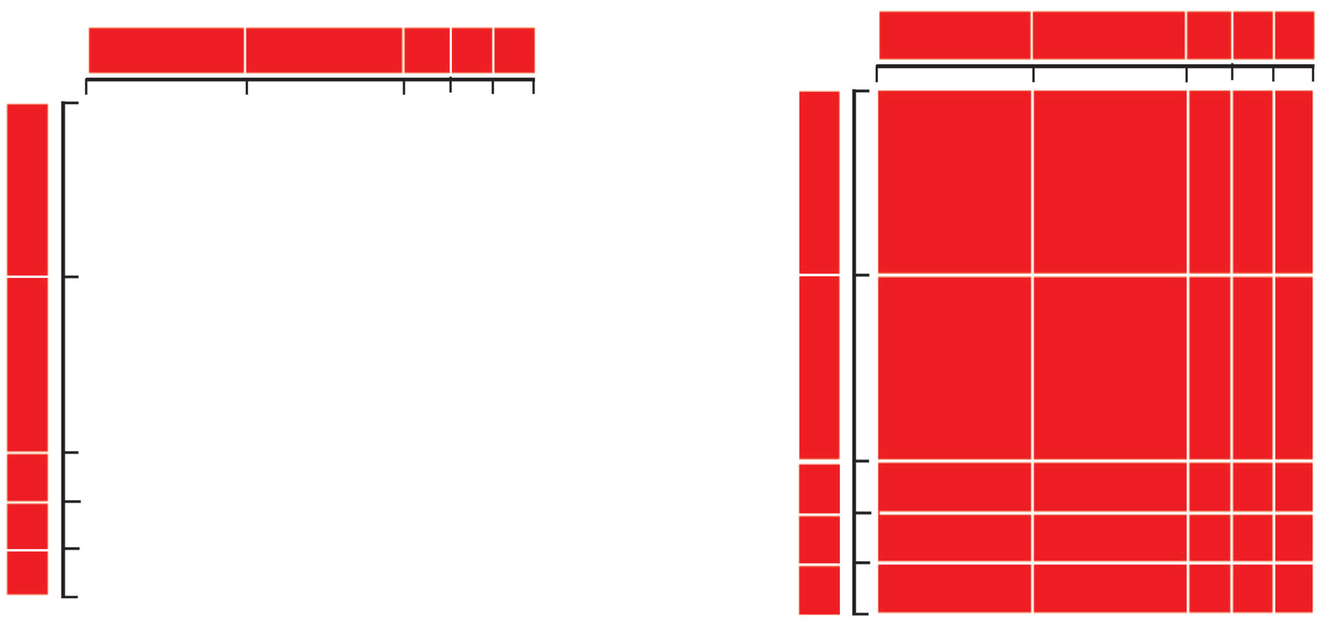
14: Multiplication of Binomials (continued)

Example 4: Find the value of $(x + 2)(2x + 1)$



Therefore $(x + 2)(2x + 1) = 2x^2 + 5x + 2$

Example 5: Find the value of $(2x + 3)^2$

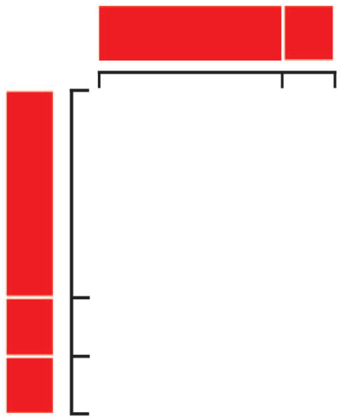
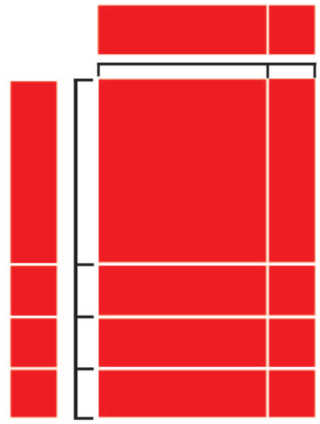
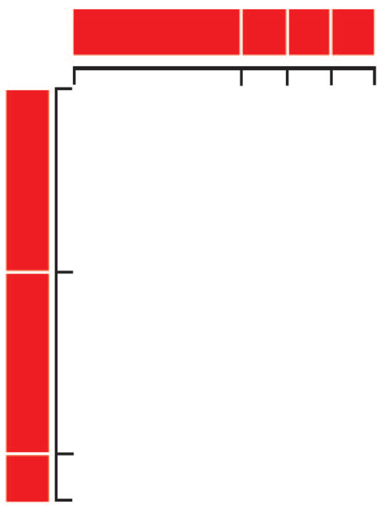
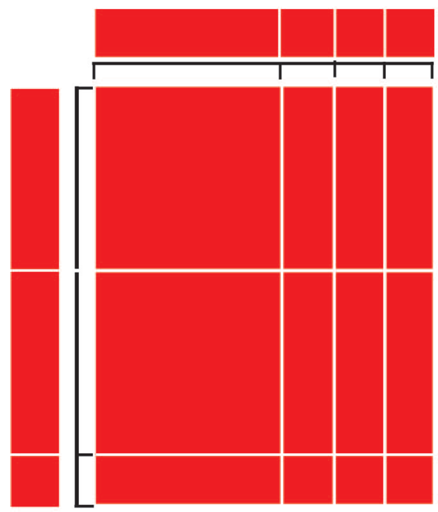


Therefore $(2x + 3)^2 = 4x^2 + 12x + 9$

Guiding Question

- How could the answer be obtained without using the tiles? (By multiplying every term in one binomial by every term in the other.)

14: Multiplication of Binomials (continued)

Symbolic Form	Dimensions of Rectangle	Pictorial Form in Tiles	Symbol Result
$(x + 2)(x + 1)$			$x^2 + 3x + 2$
$(2x + 1)\,(x + 3)$			$2x^2 + 7x + 3$

Once the students are able to use this method to multiply binonomials, groups of two or three complete assigned textbook exercises.

15: Factoring

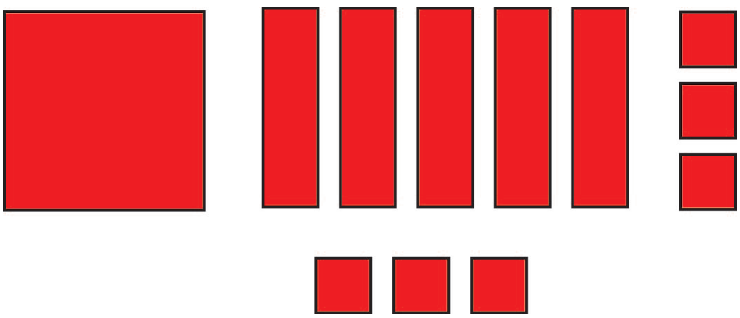
Procedure

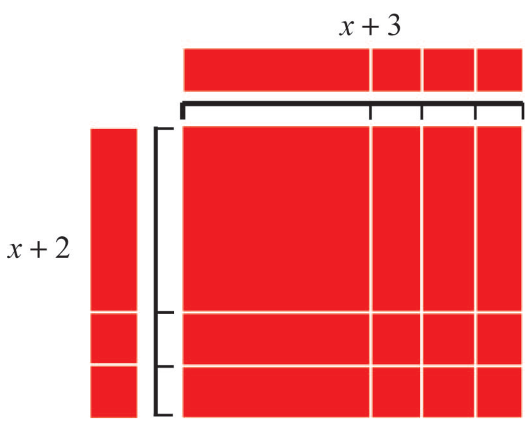
(i) Select the tiles which represent the product, i.e., area.

(ii) Make a rectangular array of the tiles by placing large square tiles in the upper left corner and the unit (1) tiles in the lower right corner.

(iii) Read the dimensions, i.e., factors, of the completed rectangle.

Example 1: Factor ($x^2 + 5x + 6$).

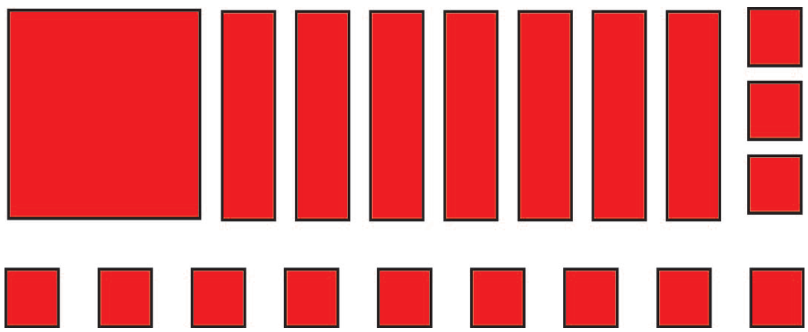


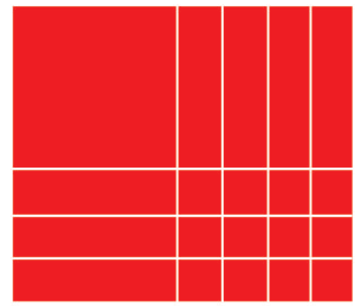


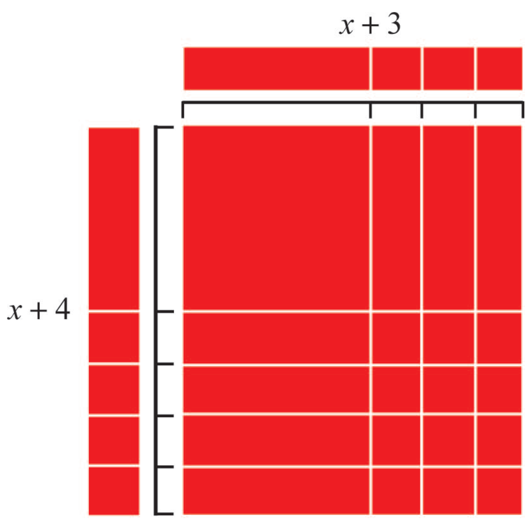
Therefore ($x^2 + 5x + 6$) factored equals ($x + 2$) ($x + 3$)

15: Factoring (continued)

Example 2: Factor $(x^2 + 7x + 12)$.

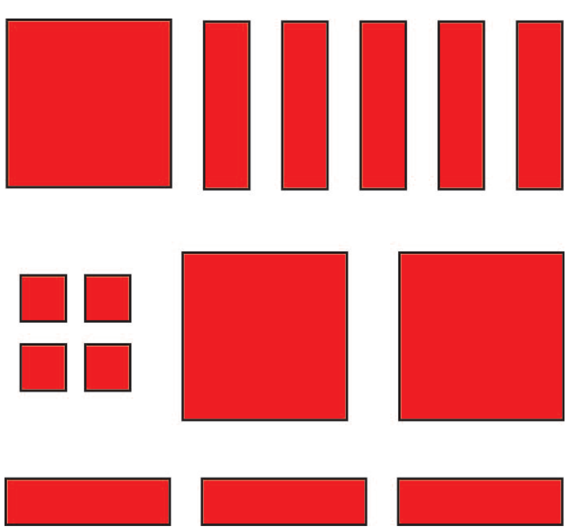






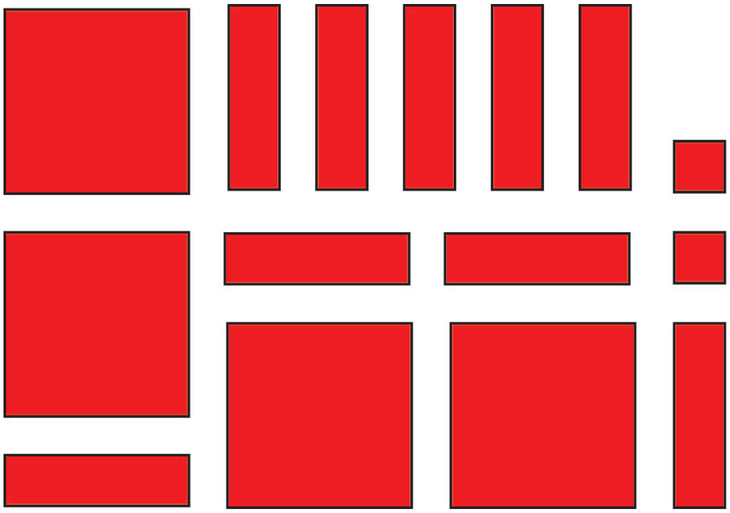
Therefore $(x^2 + 7x + 12)$ factored equals $(x + 3)(x + 4)$.

Example 3: Factor $(3x^2 + 8x + 4)$.



Therefore $(3x^2 + 8x + 4)$ factored equals $(3x + 2)(x + 2)$.

Example 4: Factor $(4x^2 + 9x + 2)$



Therefore $(4x^2 + 9x + 2)$ factored equals $(4x + 1)(x + 2)$.

Once students have become familiar with using the algebra tiles for factoring, and have an appreciation of what factoring means, explain the patterns of factoring algebraically. This can be done in either of the traditional ways of reversing the expansion process, by decomposing the middle term, or by using any of the other algebraic methods available.

Note: The algebra tiles are used only for understanding the process of factoring – not for the actual factoring of all trinomials.

Assign textbook exercises for factoring trinomials.