▲ Provincial Digital Learning Resources – What's New? Algebraic Reasoning – The Power of Visual Representations




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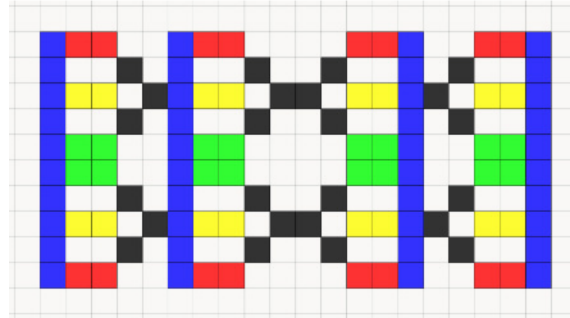
Greg (Simcoe Muskoka Catholic District School Board), Agnes (Brant Haldimand Norfolk Catholic District School Board), Markus, and Ross (Near North District School Board) are all currently working as Project Leads assigned to develop the digital resources found at mathies.ca. Greg retired as of June, so this is his last regular article with the team. Best wishes to Greg, and know you will be missed.

One of the reasons that mathies learning tools are so powerful is that they provide a means to create visual representations that can then be manipulated to solve problems and share mathematical thinking.

At the beginning of professional development sessions on mathies learning tools, we like participants to have a hands-on introduction to one of the tools. The recent mathies tools share a lot of common functionality, and an introduction to one makes participants more confident using the others. For the Colour Tiles tool, they might be asked to construct the pattern below.

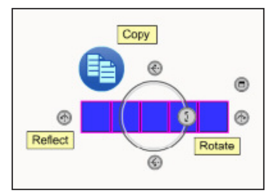


Challenge: Can you create the pattern in fewer than 25 steps?

We challenge participants to use features of the tools that allow the pattern to be built using fewer steps. For example, rather than dragging 10 blue tiles to the left edge one at a time, the multiplier button ×10 can be used to bring 10 tiles out all at once.

As participants work to create the pattern in as few steps as possible, they are encouraged to discover various features of the tool, including:

- setting the multiplier to 1, 2, 5, or 10 to drag groups of tiles to the workspace
- selecting a group of tiles
- rotating a group of tiles
- copying or reflecting a group of tiles
- seeing how to Undo/Redo steps through the actions with the tool (Participants can use this feature to count the number of steps used and see if it is less than 25.)



Participants are then asked to:

a) write an expression for the number of blue tiles
b) write an expression for the total number of tiles

Participants are encouraged to write expressions that connect to how they actually built the pattern. Expressions such as these emerge from the discussion:

Number of blue tiles is:

a) 4 x 10 (4 groups of 10)
b) 4 x 5 x 2 (4 groups of 5 doubled)
c) 5 x 2 x 4 (5, two times, quadrupled)

Total number of tiles is:

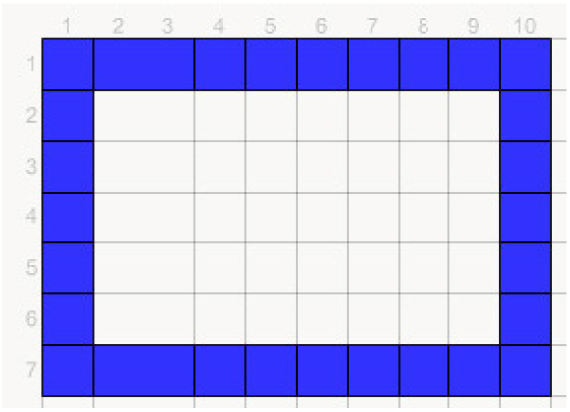
a) (5 + 3 x 2 + 3) x 2 x 2 x 2
b) (10 + 6 x 2 + 6) x 2 x 2

What does each expression reveal about how the pattern was built?

This activity not only serves as an effective introduction to the use of a mathies tool, but can lead to some interesting discussions about equivalent expressions. The visual pattern corresponding to these expressions makes them come to life in a concrete way. There is a video on the support page for Colour Tiles (see link below) that reveals one way to create this pattern.

Border Problem

In this adaptation of the border problem, the objective is to determine the number of blue tiles in this 7 by 10 rectangle, without counting each individual square.

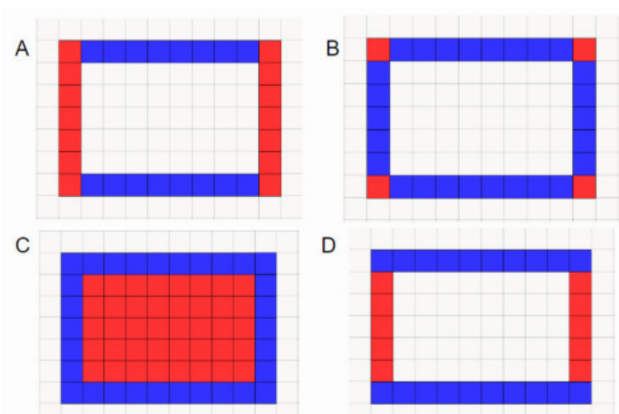


First, use the mathies Colour Tiles tool to build the rectangle.

Next use the Colour Palette to recolour the tiles to illustrate your thinking as you determine the total number of border tiles.

How can you use the various features of the tool to build this rectangle efficiently?

Does the approach to building it influence the way you determine the total number of blue tiles?



Here are some possible ways to colour the tiles.

Which image best helps you to see:

a) 2 x 10 + 2 x 5

b) 2 x 7 + 2 x 8

c) 2 x (10 + 5)

d) 4 + 2 x 5 + 2 x 8

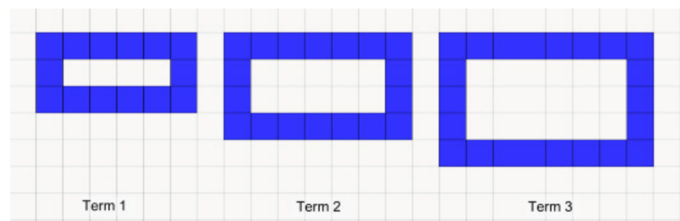
e) 10 x 7 – 8 x 5

If you created the rectangle by setting the multiplier to allow you to drag out groups of 10 tiles for the top and bottom, and groups of 5 tiles to form each side, then you might have coloured the tiles like those shown in image D and come up with 2 x 10 + 2 x 5 as the corresponding expression. Alternatively, you might have seen two "L" shapes, each made up of one blue horizontal strip and one red vertical strip, and hence come up with 2 x (10 + 5).

Each of the listed expressions corresponds to one or more of the coloured images.

Which ones might also be connected to how the rectangle was created using the tool?

Once again, there is power in seeing each of these expressions come to life visually. It facilitates making connections between the visual representation of the border and the arithmetic expressions for the number of blue tiles, which exemplifies moving from early patterning to algebraic reasoning. Mathematical processes, including making connections, representing, reasoning, and proving, are engaged by this task.

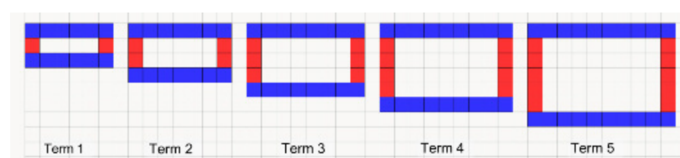


This numerical example is a good bridge to exploring more generalized algebraic expressions.

What if the 7 by 10 rectangle (shown in blue in C above) were one term in the following sequence of rectangles?

Which term would it be? Some discussion might be necessary to establish that the original rectangle is the fifth term of this sequence.

One approach is to build several terms of this sequence and colour-code them as before. Here is one possibility using the same thinking as described for image D above.



Write an expression for the number of border squares for the *n*th figure.

One of the numeric expressions that describes the fifth term of this sequence is 2 x 10 + 2 x 5. Thinking about a general term for this sequence can be addressed by considering which values in the expression remain constant and which vary? How does the coloured diagram help you to see that? In each diagram, there are two blue horizontal strips and two red vertical strips. This helps us to see that the factors of 2 are constant and leads us to generalize the expression as 2 blue strips + 2 red strips. Now, how long is each strip? How is the number of tiles in each strip related to the term number?

The number of tiles in each red vertical strip is the same as the term number.

The number of tiles in each blue horizontal strip is five more than the term number.

So, the total number of border tiles, for term n, is $2n + 2(n + 5)$.

Colouring the sequence in alternate ways, as indicated above, might result in one of the following algebraic expressions:

a) $2(n + 2) + 2(n + 3)$

b) $2n + 2(n + 3) + 4$

c) $(n + 2)(n + 5) - n(n + 3)$

d) $2(2n + 5)$

e) $2m + 2n$

Let the math discourse begin! Are these expressions equivalent? How do you know?

Do you need to use two variables, or is one sufficient? Why might it be better to use only one variable?

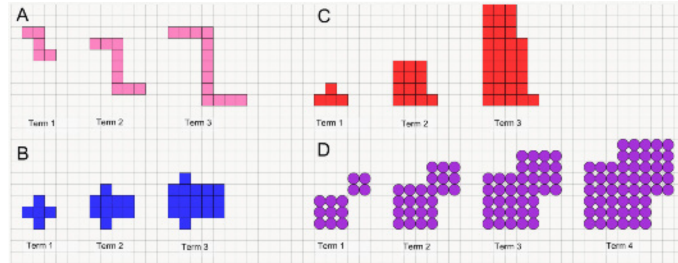
Now students have a reason to want to learn some of the basic algebraic principles, like the distributive property and collecting like terms, all with strong visual representations to support their learning.

Other Visual Patterns

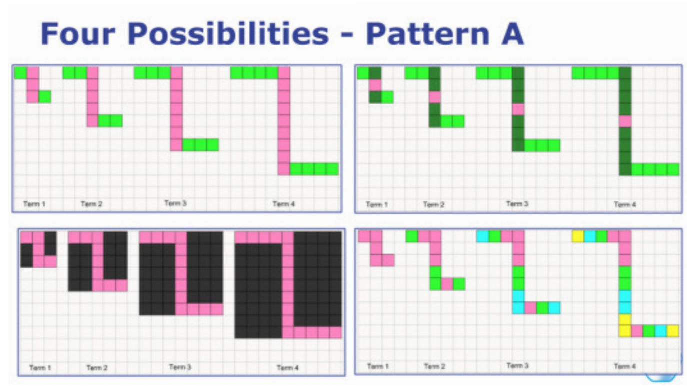
One good source of visual patterns is **www.visualpatterns.org**, started by Fawn Nguyen, one of the featured speakers at the recent OAME Annual Conference. There are over 250 patterns currently, and there is a mechanism to submit your own to the site.

Students or teachers can be given a collection of patterns and be asked to choose one (or more) for which they generate a pattern rule. They might be asked to:

1. represent the first few terms, using the Colour Tiles Tool, looking for symmetries or structures that would save them from adding tiles one by one
2. colour the tiles to make the structure explicit
3. use the annotation feature to add explanations and the pattern rule



In some of our sessions, to save time and to show many possibilities, we asked participants to match expressions to a colouring.



Which possibility is the “best” to help you see:

a) $2n + (2n + 1)$

b) $4n + 1$

c) $(2n + 1)2 - 2(n[2n])$

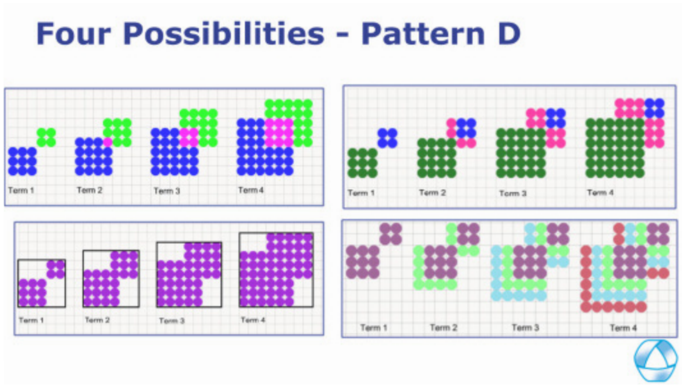
d) $5 + (n - 1)*4$

e) $2(2n) + 1$

Each of these pattern rules and colourings exploits a structural feature of the visual representation; for example, a rotational symmetry (by 180 degrees) is possible at the top right. Exploiting that rotational symmetry is possible in Colour Tiles, since it has the ability to quickly copy the tiles at the top left, rotate the copy, and move it to the bottom right.

As students practise this, they will begin to see some strategies that are mathematically useful, like using negative space (bottom left) and first differences (bottom right – where the amount added at each term is coloured differently).

As students investigate Pattern D, they might notice that its relationship will not be linear, since the number added to each term, the finite difference, changes (see bottom right below).



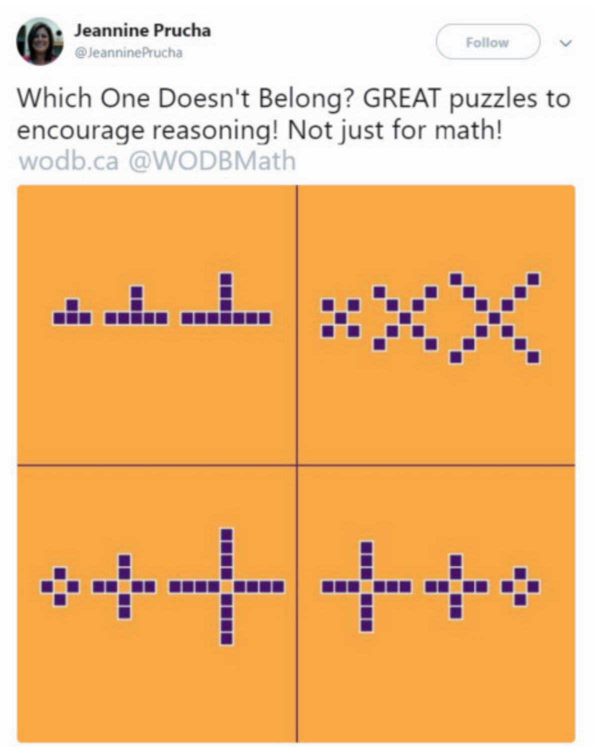
However, the use of negative space is still a very useful strategy here (bottom left). Each term is made up of a square with two 2 by 3 rectangles removed. The expression,

$$(n + 4)^2 – 12$$

matches this representation and is an expression for the pattern rule. The pattern rule is a quadratic function, and is expressed as a completion of a square in a completely visual way, without formal algebraic manipulation.

Any of these observations are best made after students (or teachers) have had ample time to construct, colour, relate quantities to the term numbers, and test various pattern rules themselves.

Another website with visual representations is **wodb.ca** (Which One Doesn't Belong?), created by Ontario's own Mary Bourassa and grown by the math community. Typically, four representations are presented, and the reader is challenged to find reasons why each one doesn't belong. This builds students' communication and creativity.



In elementary school, students can use WODB activities as part of their number talks; in fact, "Numbers" is one whole section of the site, together with "Shapes," "Graphs," and "Incomplete Sets."

It has been fun to see teachers in professional development sessions start to think about algebra in a richer way as they use visual representations. You can access mathies webinars given as part of the Renewed Mathematics Strategy at **rms.thelearningexchange.ca** and follow the links to On-Demand Professional Learning. Perhaps this article or the on-demand sessions might provide the basis for sessions that you have with some colleagues.

Feedback and Future Requests

Please feel free to send us your feedback about any mathies tool, using the Feedback Form button inside the Information Dialog, accessed from the button. Visit the support wiki page for more examples and detailed descriptions of the functionality of the tool. (With the imminent closure of wikispaces, the support pages will be migrated to **support.mathies.ca**).

You can also send your comments to **WhatsNew@oame.on.ca**. You can share your experiences on Twitter, using the hashtag #Onmathies, and follow or message us at @ONmathies. There is an increasing set of interesting posts of student and teacher work on Twitter.

To be among the first to find out about the latest digital tool developments, sign up for our email list at **mathclips.ca/WhatsNewEmailList.html**. ▲

Math decor at Humber College (OAME 2018)